86. A Characterization of Hyperplane Cuts of Smooth Complete Intersections

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In this note, we will prove the following

Theorem. *Let $M\subset \boldsymbol{P}^{N+1}$ be a smooth complete intersection. We assume for simplicity that M is non-degenerate,* i.e., *M is not contained in any linear subspace of $\boldsymbol{P}^{N+1}$. Then any hyperplane section X of M has the following two properties:*

(A) *X has only finitely many singular points;*

(B) *The Jacobian matrix J of $X\subset \boldsymbol{P}^N$ has rank $r-1$ at any singular point of X.*

Conversely, if $X\subset \boldsymbol{P}^N$ is a non-degenerate complete intersection having the properties (A) *and* (B), *then there exists a smooth complete intersection $M\subset \boldsymbol{P}^{N+1}$ such that X is a hyperplane section of M.*

Remark. The property (A) implies that X is reduced if $\dim M\geq 2$ and irreducible if $\dim M\geq 3$. Moreover, (A) is a partial refinement of the following

Zak's Theorem (see [1]). *Let $M\subset \boldsymbol{P}^{N+1}$ be an irreducible smooth non-degenerate subvariety of codimension r and X an arbitrary hyperplane section of M. Then the dimension of the singular locus of X is less than r.*

In [1], the property (A) is shown by using a suitable incidence correspondence. Our proof is more direct and elementary.

Throughout this note, we fix an algebraically closed field k of any characteristic and assume that all varieties are defined over k.

Proof of Theorem. For brevity, we introduce a symbol $V(F_1,\cdots,F_r)$ which stands for the projective variety defined by the homogeneous polynomials $F_1,\cdots,F_r$. For a given smooth complete intersection $M\subset \boldsymbol{P}^{N+1}$, we write $M=V(\tilde{F}_1,\cdots,\tilde{F}_r)$, where $\tilde{F}_i$ is a homogeneous polynomial of degree $d_i\geq 2$ in $Z_0,Z_1,\cdots,Z_{N+1}$. By a suitable linear transformation of the coordinates, we may assume that

$$X=M\cap\{Z_{N+1}=0\}=V(\tilde{F}_1,\cdots,\tilde{F}_r,Z_{N+1}).$$

Putting $F_i(Z_0,\cdots,Z_N)=\tilde{F}_i(Z_0,\cdots,Z_N,0)$, we write

$$\tilde{F}_i(Z_0,\cdots,Z_{N+1})=F_i(Z_0,\cdots,Z_N)+Z_{N+1}G_i(Z_0,\cdots,Z_{N+1}).$$

Denote by $\tilde{J}(p)$ and $J(p)$ the Jacobian matrices of the defining equations $\{\tilde{F}_1,\cdots,\tilde{F}_r\}$ and $\{F_1,\cdots,F_r\}$ at $p\in X$, respectively. Then, since $Z_{N+1}=0$ on X, we have