80. A Note on Modularity in Atomistic Lattices

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Let L be an atomistic lattice ([1], (7.1)), and let A, B be subsets of L. If (a, b) is a modular pair (resp. dual-modular pair) for every $a \in A$ and $b \in B$, we write $(A, B)M$ (resp. $(A, B)M^*$). We denote by Ω the set of atoms of L, and we put

$$\Omega^n = \{p_1 \vee \cdots \vee p_n ; p_i \in \Omega\} \qquad (n=1, 2, \cdots).$$

Evidently, $\Omega^1 = \Omega$ and $\Omega^n \subset \Omega^{n+1}$. Moreover, we put

$$F = \bigcup_{n=1}^{\infty} \Omega^n \cup \{0\}.$$

$(L, F)M$ means that L is finite-modular ([1], (9.1)), and each of $(\Omega, L)M$ and $(\Omega, L)M^*$ is equivalent to that L has the covering property ([1], (7.6)). If $A_1 \subset A_2$ and $B_1 \subset B_2$, then evidently $(A_2, B_2)M$ implies $(A_1, B_1)M$, and $(A_2, B_2)M^*$ implies $(A_1, B_1)M^*$.

In the previous paper [3], the following equivalences and non-trivial implications were proved:

(1) For any $A \subset L$, $(A, L)M \Longleftrightarrow (A, L)M^*$, $(A, F)M \Longleftrightarrow (A, F)M^*$, $(A, \Omega^n)M \Longleftrightarrow (A, \Omega^{n-1})M^*$ $(n \geq 2)$. ($(L, \Omega)M$ always holds.)

(2) $(L, F)M^* \Longrightarrow (F, L)M^*$.

(3) $(L, \Omega^n)M^* \Longleftrightarrow (L, F)M^*$ for $n \geq 1$.

(4) $(F, \Omega^n)M^* \Longleftrightarrow (F, F)M^*$ for $n \geq 1$.

(5) $(\Omega^n, F)M^* \Longleftrightarrow (F, F)M^*$ for $n \geq 2$.

(6) $(\Omega^n, \Omega)M^* \Longleftrightarrow (\Omega^{n-1}, \Omega^2)M^* \Longleftrightarrow \cdots \Longleftrightarrow (\Omega^2, \Omega^{n-1})M^*$ for $n \geq 3$.

(7) $(\Omega^2, \Omega^{n-1})M^* \Longrightarrow (\Omega, \Omega^n)M^*$ for $n \geq 2$.

Moreover, it was shown by examples that the implications (2) and (7) and the following implications are not reversible:

$$(\Omega^2, L)M^* \Longrightarrow (\Omega^2, F)M^* \Longrightarrow \cdots \Longrightarrow (\Omega^2, \Omega^n)M^* \Longrightarrow \cdots \Longrightarrow (\Omega^2, \Omega)M^*,$$

$$(\Omega, L)M^* \Longrightarrow (\Omega, F)M^* \Longrightarrow \cdots \Longrightarrow (\Omega, \Omega^n)M^* \Longrightarrow \cdots \Longrightarrow (\Omega, \Omega)M^*,$$

$$(\Omega^2, L)M^* \Longrightarrow (\Omega, L)M^*, \qquad (\Omega^2, F)M^* \Longrightarrow (\Omega, F)M^*.$$

But, it remained open whether the following implications are reversible or not:

$$(F, L)M^* \Longrightarrow \cdots \Longrightarrow (\Omega^n, L)M^* \Longrightarrow \cdots \Longrightarrow (\Omega^2, L)M^*.$$

In this paper, we shall prove that these implications are reversible, that is,

Theorem. *For an atomistic lattice L,*

(8) $(\Omega^n, L)M^* \Longleftrightarrow (F, L)M^*$ *for* $n \geq 2$.

To prove this theorem, we prepare the following lemma which