78. On Integral Transformations Associated with a Certain Riemannian Metric

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§1. Statement of the result. Let (M, g) be a complete, connected and simply connected Riemannian manifold of $\dim M=m$. We consider the following integral transformation with a parameter $t>0$.

$$(H_t f)(x)=(2\pi t)^{-m/2}\int_M \rho(x, y)e^{-d^2(x,y)/2t}f(y)d_g(y),$$

where $d_g(y)=g(y)^{1/2}dy$, $g(y)=\det g_{ij}(y)$, $d(x, y)$ denotes the Riemannian distance between x, y and $\rho(x, y)=|\det(d\,\mathrm{Exp}_x^{-1})_y|^{1/2}$ with Exp_x standing for the exponential mapping at x.

We assume the following:

(A.1) (M, g) has a non-positively pinched sectional curvature, i.e. there exists a constant $k>0$ such that for any 2-plane π, the sectional curvature K_π satisfies $-k^2\leqq K_\pi\leqq 0$.

(A.2) There exist constants C_1, C_2 such that for any x, y and $z\in M$, we have

$$|\Delta^{(z)}\rho(x, z)|\leqq C_1,$$
$$|\Delta^{(z)}\rho(x, z)-\Delta^{(z)}\rho(y, z)|\leqq C_2 d(x, y)$$

where $\Delta^{(z)}$ is the Laplace-Beltrami operator acting on a function of z, i.e.,

$$\Delta^{(z)}f(z)=g(z)^{-1/2}\sum_{i,j=1}^{m}(\partial/\partial z^i)(g(z)^{1/2}g^{ij}(z)(\partial f(z)/\partial z^j)).$$

Theorem. *Let (M, g) be a Riemannian manifold satisfying above conditions. Then, we have the following for an arbitrary number $T>0$.*

(a) *The integral transformation H_t defines a bounded linear operator in $L^2(M, d_g)$ for $0<t<T$.*

(b) $s-\lim_{t\to 0+}H_t f=f$ *for* $f\in L^2(M, d_g)$.

(c) *There exists a constant C_3 such that*

$$\|H_{t+s}f-H_tH_sf\|\leqq C_3((t+s)^{3/2}-t^{3/2}+s^{3/2})\|f\|$$

for $0<t, s, t+s<T$ and $f\in L^2(M, g)$.

(d) *There exists a limit in operator norm* $\lim_{k\to\infty}(H_{t/k})^k$ *for any $t>0$, denoted by $\boldsymbol{H}_t$, which forms with $\boldsymbol{H}_0=Id$ a C^0-semi group in*

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