76. A Uniqueness Result for the Semigroup Associated with the Hamilton-Jacobi-Bellman Operator

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1. Introduction. We consider controlled diffusion processes of the form;

$$(1)\qquad \begin{cases} dX(t)=\sigma(X(t),\ v(t))dB_t+b(X(t),\ v(t))dt \\ X(0)=x\in R^N \end{cases}$$

where B_t is an n-dimensional Brownian motion in some probability space (Ω, F, F_t, P), equipped with a filtration satisfying the usual conditions, $\sigma(x, v)$(resp. $b(x, v)$) is an $N\times n$ matrix-valued (resp. N-vector-valued) function on $R^N\times V$ and V is a separable metric space. Precise assumptions on σ, b will be made later on.

The control v is any progressively measurable process with respect to F_t taking its value in a compact subset of V. We introduce a cost function of the form:

$$(2)\qquad J(x,t,\phi,v(\cdot))=E\int_0^t f(X(s),\ v(s))\exp\left(-\int_0^s c(X(\lambda),\ v(\lambda))d\lambda\right)ds + \phi(X(t))\exp\left(-\int_0^t c(X(s),\ v(s))ds\right)$$

where $f(x, v)$, $c(x, v)$ and $\phi(x)$ are real valued functions.

We will always assume: $\exists C>0$ such that

$$(3)\qquad \begin{cases} \|D_x^\alpha\psi\|_{L^\infty(R^N)}\leq C,\ \forall v\in V,\ \forall|\alpha|\leq 2,\ \forall\psi=\sigma,\ b,\ f,\ c. \\ \psi(x,v) \text{ is continuous in } v,\ \forall x\in R^N,\ \forall\psi=\sigma,\ b,\ f,\ c. \end{cases}$$

(4) $\phi\in X=BUC(R^N)=\{v\in C_b(R^N),\ v \text{ is uniformly continuous on } R^N\}$.

Finally we set

$$(5)\qquad J(x,t,\phi)=\inf_{v(\cdot)} J(x,t,\phi,v(\cdot))$$

where the infinimum is taken over all controls $v(\cdot)$ defined above. We also denote by $(S_0(t)\phi)(x)=J(x,t,\phi)$.

Then, we know (see A. Bensoussan-J. L. Lions [2], N. V. Krylov [6], M. Nisio [11]) that the mathematical formulation of the *dynamic programming principle* is the following:

i) $S_0(t)$ is a semigroup on X.

In addition, one knows that S_0 satisfies:

ii) $J(x,t,\phi)\in BUC(R^N\times[0,T])$ $(\forall T<\infty)$ (or in other words $S_0(\cdot)$ is strongly continuous),

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