72. Cech Cohomology of Foliations and Transverse Measures

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§1. **Introduction.** For the holonomy groupoid Γ of a foliation (M, F), the transverse measure with modulus in the non-commutative integration theory [1] of A. Connes is an extension of the ordinary concept of transverse measures for the foliation. We will find in Theorem 1 a necessary and sufficient condition for the modulus δ in order that a faithful transverse measure in the Lebesgue measure class with this modulus δ exists. The condition is that δ belongs to the canonical Cech cohomology class of the foliation.

We shall define the associated foliation $(\tilde{M}, \tilde{F})$ for a given foliation (M, F) and show in Theorem 2 that the canonical Cech cohomology class of $(\tilde{M}, \tilde{F})$ vanishes and the von Neumann algebra associated with $(\tilde{M}, \tilde{F})$ is the crossed product of the same for (M, F) by its modular action.

§2. **Definitions.** We mainly follow the notations and the terminology in [1]. We assume that the holonomy groupoid Γ is a Hausdorff space.

Let us define a sheaf C_F^∞ on M, whose sections are real-valued C^∞-functions constant along leaves. More precisely, for each open subset U of M, the sections of C_F^∞ over U is given by

(1) $$C_F^\infty(U) = \{f \in C^\infty(U)\,;\, Xf = 0 \text{ for } X \in TF\}.$$

Analogously we define a sheaf L_F on M, whose sections are Lebesgue measurable functions constant along leaves. Its precise definition is as follows: Set

(2) $\tilde{L}_F(U) = \{f\,;\, f$ is a real-valued Lebesgue measurable function, constant along plates in $U\}$.

Here we don't distinguish two almost equal functions. $\tilde{L}_F(U)$ forms a presheaf on M and L_F is defined to be the sheaf generated by $\tilde{L}_F$.

Now we associate a cohomology class $c(F)$ in $H^1(M, C_F^\infty)$ with the foliation (M, F). Take an open covering $\mathcal{U} = \{U_\alpha\}$ of M by foliated charts. We denote the transversal coordinates in U_α by q_α^i. Let $c_{\alpha\beta}$ be an element of $C_F^\infty(U_\alpha \cap U_\beta)$ defined by

(3) $$c_{\alpha\beta} = \log \left| \det \left(\frac{\partial q_\beta^i}{\partial q_\alpha^j} \right) \right|.$$

Then $c_{\alpha\beta}$ satisfies the cocycle condition