48. On the Convergence of $\sum_{n=1}^{\infty} n^{-\alpha} \sin(n^{\beta}\theta)$

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0. Convergence problem of $\sum_{n=1}^{\infty} n^{-\alpha}\sin(n^{\beta}\theta)$ is decided when $0<\alpha\leqq 1$, $0\leqq\beta\leqq 1$ (cf. [3] Theorem 84), but not when $0<\alpha\leqq 1$, $1<\beta$.

From a known result relating to the Gaussian sum (cf. [2] Theorem 4.15, also [4]), we have for $\theta=2l\pi/(4m+1)$, $(2l+1)\pi/(2m+1)$

$$\sum_{k=1}^{n} \sin(k^2\theta)=O(1), \tag{1}$$

and for $\theta=(2l+1)\pi/2m$, $2l\pi/(4m+3)$

$$\sum_{k=1}^{n} \sin(k^2\theta)=Bn+O(1). \tag{2}$$

Hence for example, by partial summation

$$\sum_{n=1}^{\infty} n^{-\alpha}\sin(n^2\theta) \tag{3}$$

converges for $\theta=2l\pi/(4m+1)$*) and diverges for $\theta=(2l+1)\pi/2m$, provided $0<\alpha\leqq 1$.

On the one hand, Wilton ([9] Theorem B, cf. also [8]) showed among other things that when $0<\alpha<1$, $1<\beta\leqq 2-2\alpha$

$$\sum_{n=1}^{\infty} n^{-\alpha}\exp(in^{\beta}\theta) \qquad (i^2=-1) \tag{4}$$

diverges for all $\theta>0$.

In this paper we prove the following

Theorem. *If $\alpha>0$ and $1<\beta<2\alpha$, then* (4) *converges for all $\theta>0$.***)

1. *Proof of Theorem.* By the Euler summation formula,

$$\begin{aligned}\sum_{m=1}^{n}\frac{\sin(m^{\beta}\theta)}{m^{\alpha}} &= \frac{1}{2}\left(\sin\theta+\frac{\sin(n^{\beta}\theta)}{n^{\alpha}}\right)+\int_1^n \frac{\sin(t^{\beta}\theta)}{t^{\alpha}}\,dt \\ &\quad +\beta\theta\int_1^n\frac{\phi(t)\cos(t^{\beta}\theta)}{t^{\alpha-\beta+1}}\,dt-\alpha\int_1^n\frac{\phi(t)\sin(t^{\beta}\theta)}{t^{\alpha+1}}\,dt \\ &=\frac{1}{2}\left(\sin\theta+\frac{\sin(n^{\beta}\theta)}{n^{\alpha}}\right)+I_1^n+\beta\theta I_2^n-\alpha I_3^n, \text{ say,}\end{aligned}$$

where

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*) [6] appears to contain some incorrect arguments. It also contradicts to [9] e.g. when $0<\alpha\leqq 1/3$.

**) Note that we cannot admit $\beta=2\alpha$ for $\alpha<1/2$ in Wilton's, and for $\alpha>1/2$ in our theorem as shown by (2), (3).

In another way, we have proved that (4) converges for all $\theta>0$, provided $\alpha=1/2$, $\beta<3/2$ (cf. [1]).