# 29. *Functional Equations and Hypoellipticity*

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(Communicated by Kôsaku YOSIDA, M. J. A., March 12, 1982)

1. In this note we investigate the problem whether all the continuous or all the locally integrable solutions of certain functional equations are $C^\infty$ or not. An affirmative answer to this problem under weak regularity assumptions on the equations enables one to make easier to find all the continuous or sometimes all the locally integrable solutions of the equations. Because we can use a powerful means-differentiation (see [1], [2]). The aim of this note is to give a general method for this problem. We consider the functional equation of the unknown $f(x)$:

(1.1) $\sum_{j=1}^{k} a_j(x,t)f(h_j(x,t))=F(x, f(l_1(x)), \cdots, f(l_s(x)))+b(x,t)$

where $x\in R^n$ and $t\in R^r$, with the assumptions followed.

(A.1) $a_j(x,t),\ b(x,t)\in C^\infty(R^n)$ for every fixed $t$ from an open set $\omega\subseteq R^r$, $j=1,\cdots,k$,

(A.2) $a_j(x,t),\ b(x,t)\in C^m(R^n\times\omega)$, $j=1,\cdots,k$,

(A.3) the mappings $x\mapsto y=h_j(x,t)$ are diffeomorphisms in $R^n$ for every fixed $t\in\omega$, $j=1,\cdots,k$,

(A.4) $h_j(x,t)\in C^m(R^n\times\omega)$ and its inverse $h_j^{-1}\in C^m$, $j=1,\cdots,k$,

(A.5) $F(x,z_1,\cdots,z_s)\in C(R^{n+s})$,

(A.6) $l_j(x)\in C(R^n)$, $j=1,\cdots,s$.

A locally integrable function $f(x)$, $x\in R^n$, is said to be *a solution of* (1.1) *in the sense of distribution* (or *a distribution solution*) if

(1.2) $$\sum_{j=1}^{k}\int_{R^n} a_j(x,t)f(h_j(x,t))\phi(x)dx$$
$$=\int_{R^n} F(x,f(l_1(x)),\cdots,f(l_s(x)))\phi(x)dx+\int_{R^n} b(x,t)\phi(x)dx$$

for each $\phi\in\mathcal{D}$ and every fixed $t\in\omega$. We can write it briefly

(1.3) $$\sum_{j=1}^{k}(a_j(x,t)f(h_j(x,t)),\ \phi(x))_x$$
$$=(F(x,f(l_1(x)),\cdots,f(l_s(x))),\ \phi(x))_x+(b(x,t),\phi(x))_x.$$

Let be

$$\partial_x=\left(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_n}\right),\quad \partial_t=\left(\frac{\partial}{\partial t_1},\cdots,\frac{\partial}{\partial t_r}\right),\quad \alpha=(\alpha_1,\cdots,\alpha_n)$$
$$\partial_x^\alpha=\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n},\qquad \partial_t^\beta=\left(\frac{\partial}{\partial t_1}\right)^{\beta_1}\cdots\left(\frac{\partial}{\partial t_r}\right)^{\beta_n},$$
$$D_x^\alpha=(-i)^{|\alpha|}\partial_x^\alpha,$$

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