2. A Note on a Regularity of Irreducible Characters of a Non-Connected Lie Group

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1. Let G be a Lie group. We call a character of an irreducible unitary representation (IUR) of G an irreducible character of G. Little is studied about the regularity of irreducible characters when G is not connected. In this note, we present an interesting example from this point of view.

First recall some known facts. Let G be reductive, and assume that $\mathrm{Ad}(G)$ is contained in the connected complex adjoint group of $\mathfrak{g}_c$, where $\mathfrak{g}_c$ denotes the complexification of the Lie algebra $\mathfrak{g}$ of G. Then any irreducible character is a distribution on G which coincides with a locally summable function on G, analytic on an open dense subset of G (see [2, p. 132]). On the other hand, Shintani studied in [4] the relation between IURs of the group G generated by $G^0=SL(2, \boldsymbol{C})$ and σ, and those of $G_\sigma=SL(2, \boldsymbol{R})$, where σ is the complex conjugation of matrices. In this case, $G=G^0 \cup G^0\sigma$, and $\mathrm{Ad}(\sigma)$ is an outer automorphism of $\mathfrak{g}_c$. Hence, to use the explicit forms of irreducible characters of G and those of G_σ, as an essential tool, he had to establish for G the similar result as above [4].

Now let G be nilpotent and connected. Then the character of an infinite-dimensional IUR is a distribution on G supported by a subvariety of lower dimension. In fact, by Kirillov's orbit method, such a representation is equivalent to that induced from a proper connected subgroup H by a unitary character of it. Therefore, the character of the representation is supported by the closure $\overline{G(H)}$ of the union $G(H)$ of gHg^{-1} over $g \in G$. Since H is proper, the normalizer N_G of H in G is strictly bigger than H. Put $\bar{g}(H)=gHg^{-1}$ for $\bar{g}=gN_G$, then $G(H)$ is the union of $\bar{g}(H)$ over $\bar{g} \in G/N_G$. Note that N_G is connected, and use the structure of a connected nilpotent Lie group given in [3, p. 83], then we see that $\dim \overline{G(H)}=\dim H+\dim (G/N_G)<\dim G$.

In these connections, we are interested in the following problems. Assume that G is unimodular, and that any irreducible character is a distribution on G, that is, for any IUR T of G, the operator

$$T(\varphi)=\int_G \varphi(g)T(g)dg \qquad \text{for } \varphi \in C_0^\infty(G)$$

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