94. A Remark on Certain Stochastic Control Problem

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We consider the following problem of stochastic control in which the system is given by the Ito-type stochastic differential equation: $dX_t=\psi(t, X_t)dt+dB_t$, where ψ is a bounded measurable function, and the cost function to be minimize with respect to ψ is of the form:

$$E\left[\int_s^T L(t, |X_t|)\,dt+h(|X_T|)\right].$$

Our aim is to obtain an explicit form of an optimal control. In addition, as corollary, we can show the existence of solutions of certain partially differential equations of parabolic type with singular drift coefficients.

§ 1. **Representation of optimal control.** Let T be a fixed positive time and assume that $0\leqq s\leqq T$. Consider the following d-dimensional stochastic differential equation:

$$\text{(1.1)}\qquad \begin{cases} dX_t=\psi(t, X_t)dt+dB_t, & s\leqq t<T \\ X_s=x, \end{cases}$$

where (B_t), $0\leqq t\leqq T$, is a Brownian motion started from 0, ψ is a Borel function from $[0, T]\times R^d$ to R^d such that $\sup_{s,x}|\psi(s, x)|\leqq 1$, and x is a vector (fixed) in R^d. By Ψ we mean the class of such ψ's and any element of Ψ is called *an admissible control.* For any $s\in[0, T)$, $\Psi(s)$ stands for the restriction of Ψ on $[s, T]\times R^d$. Let us note that for all $\psi\in\Psi(s)$ there exists a unique strong solution of Eq. (1.1) in pathwise sense ([4]). By $X_t^{s,x,\psi}$ we mean the solution of Eq. (1.1) associated with $\psi\in\Psi(s)$. The cost function associated with ψ is given by the formula:

$$\text{(1.2)}\qquad J(s, x, \psi)=E\left[\int_s^T L(t, X_t^{s,x,\psi})dt+h(X_T^{s,x,\psi})\right]$$

where $L(s, x)$ and $h(x)$ are Borel functions from $[0, T]\times R^d$ to R_+ and from R^d to R_+ respectively. An element $\psi^0\in\Psi$ is called *optimal* if ψ^0 satisfies the relation:

$$\text{(1.3)}\qquad J(s, x, \psi^0)=\inf_{\psi\in\Psi(s)} J(s, x, \psi),\qquad \text{for all } (s, x).$$

Now we need the following notations: $Q^0=(0, T)\times R^d$ and $\bar{Q}^0=[0, T]\times R^d$; $C^j(R^d)$ is the class of functions with continuous partially derivatives of all orders $\leqq j$ on R^d; for $Q\subset R^{d+1}$, $C^{1,2}(Q)$ means the set of $\phi(t, x)$ with $\partial\phi/\partial t$, $\partial\phi/\partial x_i$, $\partial^2\phi/\partial x_i\partial x_j$, $i, j=1, \cdots, d$, continuous on Q; $C_p^{1,2}(Q)$ is the class of $\phi\in C^{1,2}(Q)$ such that $|\phi(t, x)|\leqq c(1+|x|)^k$ for all $(t, x)\in Q$, where c and k are constants not depending upon (t, x), when