83. A Remark on the Boundary Behavior of Quasiconformal Mappings and the Classification of Riemann Surfaces

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1. Generally, a property of open Riemann surfaces is not always preserved by a quasiconformal mapping. For example, the class O_{AB}, the class of Riemann surfaces on which there exists no non-constant bounded analytic function, is not quasicomformally invariant (cf. [1], [3]). In this paper, we shall study properties of Riemann surfaces which are not preserved by quasiconformal mappings.

Let R_1, R_2 be open Riemann surfaces and $f: R_1 \to R_2$ be a quasiconformal mapping. The main purpose of this paper is to construct the counter examples for the following problems.

I. Suppose that R_j $(j=1,2)$ are hyperbolic, that is, R_j have Green's functions $g_j(\cdot, p_j)$ with poles at $p_j \in R_j$. Are the Green's functions *quasi-invariant*? Precisely, dose the following inequality

$$g_1(z, p_1) \leqq M g_2(f(z), f(p_1))$$

hold for any point z on R_1 and a constant $M(>0)$ not depending on z?

II. Suppose R_1 is in *Widom class* (cf. [5]), that is, R_1 is hyperbolic and for each point $p_1 \in R_1$,

$$\int_0^\infty \beta(t: p_1)dt < +\infty,$$

where $\beta(t: p_1)$ is the first Betti number of $\{p \in R_1 : g_1(p, p_1) > t\}$. Is R_2 also in *Widom class*?

III. Let R_1 and R_2 be not in O_{AB}. Suppose that R_1 is *AB-separable*, that is, for any points $p, q \in R_1$ $(p \neq q)$ there is a bounded analytic function g such that $g(p) \neq g(q)$. Is R_2 also *AB-separable*?

Finally in § 4, we shall give a theorem concerning with Problems II and III.

2. First of all, we recall the following proposition due to A. Beurling and L. Ahlfors (cf. [1], [2]).

Proposition. *There exists a quasiconformal automorphism of the upper half plane with the boundary function $h(x)$ $(x \in \boldsymbol{R})$ if and only if*

$$\rho^{-1} \leqq \frac{h(x+t)-h(x)}{h(x)-h(x-t)} \leqq \rho \tag{1}$$

for some constant $\rho \geqq 1$ and for all x and $t(\neq 0)$.