7. *Scattering Techniques in Transmutation and some Connection Formulas for Special Functions*

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1. Introduction. Fadeev in [11] develops a technique for displaying certain operators of interest in scattering theory in terms of transmutations; this allows one to give an essentially unified derivation of the Gelfand-Levitan and Marčenko equations (which is generalized in Carroll [6]). In particular the link between the Gelfand-Levitan and Marčenko equations is shown in [11] to be a certain transmutation operator $\tilde{U}$ and in this article we determine the natural generalization $\tilde{\mathcal{B}}$ (or $\tilde{\mathcal{B}}$) of $\tilde{U}$ in the transmutation framework of Carroll [2]–[5]; then, in a context based on harmonic analysis in rank one noncompact symmetric spaces, we show how the use of such operators $\tilde{\mathcal{B}}$ provides a transmutation meaning and abstract derivation for various types of formulas connecting special functions with integrals of Riemann-Liouville and Weyl type (cf. Flensted-Jensen [12], Koornwinder [13], Askey-Fitch [1], Chao [8]). One particular feature of $\tilde{U}$ which relates Riemann-Liouville and Weyl type integrals in the relation $\tilde{U}=(U^{-1})^*$ for a basic transmutation operator U and this provides complementary types of triangular kernels (cf. here Erdélyi [10] for a related use of adjointness). In our more general framework adjointness plays a different role but we obtain similar triangularity results for the analogous $\mathcal{B}$ and $\tilde{\mathcal{B}}$ by other methods (Theorem 2.1). The details will appear in [7].

2. Basic constructions. We will work with differential operators of the form $P(D)u=(Au')'/A$ where $A(x)$ will have properties modeled on $P(D)$ being the radial Laplace-Beltrami operator on a noncompact Riemannian symmetric space of rank one (cf. [9], [12], [13] for details). Let $\varphi_\lambda^P(t)$ be a "spherical function" satisfying $P(D)\varphi_\lambda^P=(-\lambda^2-\rho^2)\varphi_\lambda^P$, $\varphi_\lambda^P(0)=1$, and $D_t\varphi_\lambda^P(0)=0$, where $\rho=\lim(1/2)A'/A$ at $t\to\infty$. Thus $\varphi_\lambda(t)=\varphi_\lambda^P(t)\sim H(t,\mu)$ for $\mu=-\lambda^2$ and $\hat{P}=P+\rho^2$ (notation of [2]–[5]). We set $\Omega(x,\mu)=\Omega_\lambda(x)=\Omega_\lambda^P(x)=\Delta_P(x)\varphi_\lambda^P(x)$ where $\Delta_P(x)=A(x)$ for $P(D)$. Then $\hat{P}^*(D)\Omega_\lambda^P=\mu\Omega_\lambda^P$ where $P^*(D)\psi=[A(\psi/A)']'$ denotes the formal adjoint of $P(D)$. A typical example of $\Delta_P(x)$ here is $\Delta_P(x)=\Delta_{\alpha\beta}(x)=(e^x-e^{-x})^{2\alpha+1}(e^x+e^{-x})^{2\beta+1}$ with $\rho=\alpha+\beta+1$ in which

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