37. A Further Generalization of the Ostrowski Theorem in Banach Spaces

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§ 1. Let $f: D\subset R^n\to R^n$ be Fréchet differentiable at an interior point x^* of D and $f(x^*)=x^*$. If the spectral radius of $f'(x^*)$ satisfies $\rho(f'(x^*))<1$, then x^* is a point of attraction (or an attractor) of the iterates $f(x_k)=x_{k+1}$, i.e., there is an open neighborhood S of x^* such that $S\subset D$ and, for any $x_0\in S$, the iterates $\{x_k\}$ defined by $f(x_k)=x_{k+1}$ all lie in D and converge to x^*. The sufficiency of $\rho(f'(x^*))<1$ for a point of attraction was proved by Ostrowski [4, pp. 118–120] (first edition) under somewhat more stringent condition on f, and later by Ostrowski [4, pp. 161–164] (second edition) and [5, pp. 150–152] under those of the above theorem. Using the well known spectral radius formula in Banach algebra, Kitchen [3] extended Ostrowski's theorem to an arbitrary Banach space. Ostrowski's theorem occupies a special place in the study of Newton's iteration processes [4]. To study non-stationary (nonautonomous) processes and Newton-SOR processes, Ortega and Rheinboldt [4, pp. 349–350] extended Ostrowski's theorem in a more general form. Generalizing further, we shall extend this general form to an arbitrary Banach space.

§ 2. Let X and Y be two real Banach spaces. A family of maps $\{f_h\}$, where $f_h: D\subset X\to X$ and the parameter vector h varies over some set $D_h\subset Y$, is uniformly Fréchet differentiable at an interior point of D if each f_h is Fréchet differentiable at an interior point of D if each f_h is Fréchet differentiable at x and if for any $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon)>0$, independent of h, such that $S(x,\delta)=\{y\in X: \|y-x\|<\delta\}\subset D$ and

$$\|f_h(y)-f_h(x)-f_h'(x)(y-x)\|\leqq\varepsilon\|y-x\|$$

for all $y\in S(x,\delta)$ and for all $h\in D_h$.

Theorem (Generalized Ostrowski theorem in Banach spaces). *Let X and Y be two real Banach spaces. For $f: D\times D_h\subset X\times Y\to X$ and x^* is an interior point of D such that $x^*=f(x^*, h)$ for all $h\in D_h$, assume that the family of maps $\{f_h\}$, where*

$$f_h: D\subset X\to X,\ f_h(x)=f(x,h),\ x\in D,\ h\in D_h,$$

is uniformly Fréchet differentiable at x^* for all $h\in D_h$, and that

$$f_h'(x^*)=H^{q(h)}, \qquad \text{for all } h\in D_h,$$

where H is a bounded linear operator on X satisfies $\rho(H)<1$ and $q(h)$