21. *A Solution to a Problem on the Asymptotic Behavior of Nonexpansive Mappings and Semigroups*[*)]

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Let C be a closed convex subset of a Banach space $E, T: C \to C$ a nonexpansive ($|Tx-Ty| \leqslant |x-y|$ for all x and y in C) mapping, and $S: [0, \infty) \times C \to C$ a nonexpansive nonlinear semigroup. Assume that the norm of E is uniformly Gâteaux differentiable (UG), and that the norm of its dual E^* is Fréchet differentiable (F). It was shown in [5] and [7] that if C is a (sunny) nonexpansive retract of E, then the strong $\lim_{n\to\infty} T^n x/n$ and $\lim_{t\to\infty} S(t)x/t$ exist for each x in C. However, the question whether this is true for arbitrary closed convex subsets of E has remained open [6, Problem 7] and [8, Problem 4]. The purpose of this note is to present a positive solution to this problem (Theorems 2 and 3). Theorem 1 provides a (partial) positive answer to a question of Pazy [4, p. 239].

Recall that a subset A of $E \times E$ with domain $D(A)$ and range $R(A)$ is said to be accretive if $|x_1-x_2| \leqslant |x_1-x_2+r(y_1-y_2)|$ for all $[x_i, y_i] \in A$, $i=1,2$, and $r>0$. The resolvent $J_r: R(I+rA) \to D(A)$ and the Yosida approximation $A_r: R(I+rA) \to R(A)$ are defined by $J_r=(I+rA)^{-1}$ and $A_r=(I-J_r)/r$ respectively. We denote the closure of a subset D of E by $cl(D)$ and its closed convex hull by $clco(D)$. The distance between a point x in E and D is denoted by $d(x, D)$. We shall say that D has the minimum property [4] if $d(0, clco(D))=d(0, D)$. Let J denote the duality map from E to E^*.

Theorem 1. *Let E be a Banach space with a uniformly Gâteaux differentiable norm, and let $A \subset E \times E$ be an accretive operator. If $R(I+rA) \supset cl(D(A))$ for all $r>0$, then $cl(R(A))$ has the minimum property.*

Proof. Let $x \in cl(D(A))$, $z \in Ay$, and $t>0$. Since A is accretive, $(z-A_t x, J((y-J_t x)/t)) \geqslant 0$ for all t. Let a subset of $j_t = J((y-J_t x/t)$ converge weak-star to j as $t \to \infty$. Since we always have $\lim_{t\to\infty} |J_t x/t| = d(0, R(A)) = d$ (see the proof of [9, Proposition 5.2]), we see that $|j| \leqslant \liminf_{t\to\infty} |j_t| = d$. We also have $\lim_{t\to\infty} (A_t x, J((y-J_t x)/t)) = d^2$. Therefore $(z, j) \geqslant d^2$. Since E is (UG), j does not depend on y and z. Thus

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