20. A Note on Asymptotic Strong Convergence of Nonlinear Contraction Semigroups

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1. Let H be a real Hilbert space with inner product $(\ ,\ )$, φ a proper l.s.c. (lower semicontinuous) convex functional on H into $(-\infty, \infty]$ such that the effective domain contains 0, and let $\partial\varphi$ be the subdifferential of φ. The operator $\partial\varphi$ generates a contraction semigroup, say $\{S(t)\}$, and $u(t)=S(t)x$ is an absolutely continuous solution of the initial-value problem

$$(1) \qquad \begin{cases} \dfrac{du}{dt} \in -\partial\varphi(u(t)) & \text{a.e.}\, t \in (0, \infty), \\ u(0)=x \in \overline{D(\varphi)}, \end{cases}$$

where $D(\varphi)=\{x\,;\,\varphi(x)<\infty\}$.

Recently R. Bruck [2] has treated the asymptotic strong convergence of solutions to the initial-value problem for (1) under the assumption that φ is even, i.e., $\varphi(x)=\varphi(-x)$ for $x \in D(\varphi)$. In this note we show that his approach also works for a more general case than that of even convex functionals.

Our result is stated as follows:

Theorem. *If there is a positive number α such that*

$$(2) \qquad \varphi(x)-\varphi(0) \geqq \alpha\{\varphi(-x)-\varphi(0)\} \qquad \textit{for } x \in D(\varphi),$$

then the solution $u(t)$ of the equation (1) *converges strongly as $t\to\infty$ to some minimum point of φ. That is, $s-\lim_{t\to\infty} u(t) \in F=\{x \in H : \varphi(x) =\inf \varphi\}$.*

A functional φ is even iff the inequality (2) holds for $\alpha=1$; hence our result extends Theorem 5 of Bruck [2, p. 23]. Although the proof of the above theorem is obtained by the method due to Bruck, condition (2) is considerably weaker than the assumption that φ is even.

2. **Proof of Theorem.** If $\alpha>1$, condition (2) implies that φ is constant. So, in what follows, we assume that $0<\alpha\leqq 1$. Moreover, we may assume

$$\varphi(0)=0,$$

since we have trivially $\partial\varphi=\partial(\varphi+\text{const.})$. Then, we have

$$(3) \qquad \varphi(x) \geqq \alpha\varphi(-x)=\alpha\varphi(-x)+(1-\alpha)\varphi(0) \geqq \varphi(-\alpha x)$$

for every $x \in D(\varphi)$.

The origin 0 is a minimum point of φ. In fact, for each $x \in D(\varphi)$,