100. *A Remark Concerning the Extensions of Some Group C*-Algebras*

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We investigate the extensions of the enveloping group C^*-algebras of discrete groups and show that to the free product of groups corresponds the direct sum of EXTs. As a consequence, it will be seen that the EXT of the enveloping group C^*-algebra of a free group F_n is Z^n, a result announced in L. G. Brown [2].

Let $G_k (k \in N)$ be groups, then we denote by $G_1 * G_2$ (resp. $\prod^*_{k\in N} G_k$) the free product of G_1 and G_2 (resp. $\{G_k\}_{k\in N}$). If F is a group and φ_k is homomorphism of G_k into F, then there exists a unique homomorphism φ of $\prod^* G_k$ into F such that $\varphi \circ \iota_i = \varphi_i$ for all i, where ι_i is the canonical inclusion of G_i into $\prod^* G_k$. Throughout the paper, we assume that the groups are countable, $C^*(G)$ is then separable and has a unit, where $C^*(G)$ denotes the enveloping group C^*-algebra of G.

H is a separable infinite dimensional Hilbert space, $Q(H)$ is the Calkin algebra on H, and π is the quotient map from the total operator algebra $B(H)$ onto $Q(H)$. An extension τ of $K(H)$, the algebra of compact operators, by a unital separable C^*-algebra A is a unital *-isomorphism of A into $Q(H)$. $EXT(A)$ is the family of all equivalence classes of extensions by A. Concerning these, we follow mainly the expositions in [1].

Let φ be a unital *-homomorphism of A into another unital separable C^*-algebra B. φ induces a homomorphism $\varphi^\#$ of $EXT(B)$ into $EXT(A)$ in the following way. For $[\tau] \in EXT(B)$, $\varphi^\#[\tau] = [\tau \circ \varphi \oplus \tau_0]$, where τ_0 is the trivial extension of A, the extension which comes from a unital *-isomorphism of A into $B(H)$. This is well-defined because of the equivalence of all trivial extensions.

For short, we write $EXT[G]$ in place of $EXT(C^*(G))$.

Theorem. *Let G_k be discrete groups $(k \in N)$. If $EXT[G_k]$ are groups for all k, then $EXT[\prod^*_{k\in N} G_k]$ is a group. Moreover*

$$EXT[\textstyle\prod^*_{k\in N} G_k] = \prod_{k\in N} EXT[G_k].$$

Proof. If G is a discrete group, $C^*(G)$ is generated by $\{U_g ; g \in G\}$, where U_g is the corresponding unitary to $g \in G$ in its universal representation. The canonical injection ι_i of G_i into $\prod^* G_k$ induces a *-homomorphism $\iota_{i\#}$ of $C^*(G_i)$ into $C^*(\prod^* G_k)$. $\iota_{i\#}$ also induces a homo-