69. A Note on the Tate Conjecture for K3 Surfaces

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This note discusses the openness of the image of the Galois group in the second ℓ-adic cohomology of a $K3$ surface with large Picard number defined over an algebraic number field. Especially, we prove the Tate conjecture for a $K3$ surface, whose Picard number is 20 or 19.

Let X be a smooth projective geometrically irreducible surface defined over an algebraic number field k, which satisfies the conditions:

$$\Omega^2_{X/k}=\mathcal{O}_X \quad \text{and} \quad H^1(X, \mathcal{O}_X)=0.$$

Such a surface is called a $K3$ surface ([12]). The Picard number ρ of X is defined by

$$\rho=\dim_{\boldsymbol{Q}} NS(X\otimes\bar{k})\otimes_{\boldsymbol{Z}} \boldsymbol{Q},$$

where $\bar{k}$ is the algebraic closure of k, and $NS(X\otimes\bar{k})$ is the Néron-Severi group of $X\otimes\bar{k}$. For any embedding of the field $\sigma: k\hookrightarrow \boldsymbol{C}$, put

$$\rho_\sigma=\dim_{\boldsymbol{Q}} NS(X\otimes_{k,\sigma} \boldsymbol{C})\otimes_{\boldsymbol{Z}} \boldsymbol{Q}.$$

Then the equality $\rho=\rho_\sigma$ holds.

The Betti numbers of X are given by

$$b_0=b_4=1, \quad b_1=b_3=0, \quad b_2=22.$$

Put $\rho_k=\dim_{\boldsymbol{Q}} NS(X)\otimes_{\boldsymbol{Z}} \boldsymbol{Q}$, and assume that $\rho_k=\rho$. We call $\lambda=b_2-\rho$ the Lefschetz number of X, which is the number of transcendental cycles independent modulo algebraic cycles.

Now let us recall the Brauer group $\mathrm{Br}(X\otimes\bar{k})$ of $X\otimes\bar{k}$. By Grothendieck [1], it is known to be a torsion group, and the Tate module $T_\ell(\mathrm{Br}(X\otimes\bar{k}))$ is given by the exact sequence of $\mathrm{Gal}(\bar{k}/k)$-modules

$$0\longrightarrow NS(X)\otimes \boldsymbol{Z}_\ell\longrightarrow H^2_{\text{ét}}(X\otimes\bar{k}, \boldsymbol{Z}_\ell[1])\longrightarrow T_\ell(\mathrm{Br}(X\otimes\bar{k}))\longrightarrow 0.$$

Here $\boldsymbol{Z}_\ell[1]$ is the Tate twist.

Put $V_\ell=T_\ell\otimes_{\boldsymbol{Z}_\ell} \boldsymbol{Q}_\ell$. The intersection form on $H^2_{\text{ét}}(X\otimes\bar{k}, \boldsymbol{Q}_\ell)$ is a symmetric bilinear form with values in $\boldsymbol{Q}_\ell[-2]$. We denote by $V_\ell(T)$ the orthogonal complement of $NS(X)\otimes_{\boldsymbol{Z}} \boldsymbol{Q}_\ell[-1]$. Then the restriction of the intersection form to $V_\ell(T)$ defines a non-degenerate bilinear form with values in $\boldsymbol{Q}_\ell[-2]$, and the above exact sequence induces an isomorphism of the ℓ-adic representations of $\mathrm{Gal}(\bar{k}/k)$:

$$V_\ell(T)[1]\xrightarrow{\sim} V_\ell(\mathrm{Br}(X\otimes\bar{k})).$$

Let us consider the ℓ-adic representation

$$\rho_{T,\ell}: \mathrm{Gal}(\bar{k}/k)\longrightarrow \mathrm{Aut}(V_\ell(T)).$$

By definition, $\lambda=b_2-\rho=\dim_{\boldsymbol{Q}_\ell} V_\ell(T)$. Since the characteristic of k is