57. *On the Strong Convergence of the Cèsaro Means of Contractions in Banach Spaces*

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1. **Introduction.** Throughout this paper X denotes a uniformly convex Banach space and C is a nonempty closed convex subset of X. A mapping $T: C \to C$ is called a contraction on C, or $T \in \mathrm{Cont}(C)$ if $\|Tx-Ty\| \leqq \|x-y\|$ for every $x, y \in C$. A family $\{T(t); t \geqq 0\}$ of mappings from C into itself is called a contraction semi-group on C if $T(0) = I$ (the identity on C), $T(t+s)=T(t)T(s)$, $T(t) \in \mathrm{Cont}(C)$ for $t, s \geqq 0$ and $\lim_{t\to 0+} T(t)x = x$ for every $x \in C$. The set of fixed points of a mapping T will be denoted by $F(T)$.

The purpose of this paper is to prove the following (nonlinear) mean ergodic theorems.

Theorem 1. *Let $T \in \mathrm{Cont}(C)$, $x \in C$ and $F(T) \neq \emptyset$. If $\lim_{n\to\infty} \|T^n x - T^{n+i}x\|$ exists uniformly in $i=1, 2, \cdots$, then there exists an element $y \in F(T)$ such that*

$$(\textit{the strong limit}) \lim_{n\to\infty} n^{-1} \sum_{i=0}^{n-1} T^{i+k}x = y \quad \textit{uniformly in } k=0, 1, 2, \cdots.$$

Theorem 2. *Let $\{T(t); t \geqq 0\}$ be a contraction semi-group on C, $x \in C$ and $\bigcap_{t>0} F(T(t)) \neq \emptyset$. If $\lim_{t\to\infty} \|T(t)x - T(t+h)x\|$ exists uniformly in $h>0$, then there exists an element $y \in \bigcap_{t>0} F(T(t))$ such that*

$$\lim_{t\to\infty} t^{-1} \int_0^t T(s+h)x \, ds = y \qquad \textit{uniformly in } h \geqq 0.$$

These results have been known in Hilbert space (cf. [1, 2]).

2. **Proofs of Theorems.** For a given $T \in \mathrm{Cont}(C)$ we set $S_n = n^{-1}(I+T+\cdots+T^{n-1})$ for every $n \geqq 1$. We start with the following

Lemma 1. *Let $T \in \mathrm{Cont}(C)$, $x \in C$ and $F(T) \neq \emptyset$. Suppose that*

(*) $\lim_{n\to\infty} \|T^n x - T^{n+i}x\|$ *exists uniformly in $i=1, 2, \cdots$.*

Then we have

(1) $\lim_{n,m\to\infty} \|2^{-1}(S_n T^{l+n}x + S_m T^{l+m}x) - T^l(2^{-1}S_n T^n x + 2^{-1} S_m T^m x)\| = 0$ *uniformly in $l=1, 2, \cdots$. In particular,*

(2) $\lim_{n\to\infty} \|S_n T^{l+n}x - T^l S_n T^n x\| = 0$ *uniformly in $l=1, 2, \cdots$.*

Proof. Take an $f \in F(T)$ and an $r>0$ with $r \geqq \|x-f\|$, and set $D=\{z \in X; \|z-f\| \leqq r\} \cap C$ and $U = T|_D$ (the restriction of T to D). Since D is bounded closed convex and $U \in \mathrm{Cont}(D)$, by virtue of [4, Theorem

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