44. On Formal Analytic Poincaré Lemma*)

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1. Introduction. Let X be a complex manifold and Y an analytic subset of X. Let $\Omega_X^{\cdot}$ be the complex of sheaves of germs of holomorphic forms on X and $\hat{\Omega}_X^{\cdot}$ the formal completion of $\Omega_X^{\cdot}$ along Y (cf. [4]). Then the formal analytic Poincaré lemma says that $\hat{\Omega}_X^{\cdot}$ gives a resolution of C_Y with respect to the natural augmentation $C_Y \to \hat{\Omega}_X^{\cdot}$. This was first shown by Hartshorne [4] and Sasakura [5] independently. Actually, Sasakura obtained a stronger result using his theory of stratifying analytic sets and of cohomology with growth conditions [5]. In the present note we shall give a simple alternative proof of his result using resolution, based on the idea of Bloom (cf. [2, 3.1]).

2. Statement of the result. Let $U=X-Y$ and $j: U \to X$ be the inclusion. Let I be any coherent sheaf of ideals of O_X with supp O_X/I $=Y$ where supp denotes the support. We call an open subset V of X *good with respect to* Y if V is Stein, its closure $\bar{V}$ is a Stein compact, and if the restriction map $j^*: H^i(V, C) \to H^i(Y \cap V, C)$ are isomorphic for all i, or equivalently, $H^i_{\Psi}(V-V\cap Y, C)=0$ for all i, where Ψ is the family of supports consisting of closed subsets of V which are contained in $V-V\cap Y$. In what follows for a rational number r we denote by $[r]$ the largest integer which is not greater than r, and then we write $[r]_+ = \max([r], 0)$.

Theorem. *Let V be an open subset of X which is good with respect to Y. Then there exist rational numbers c_1, c_2 with $c_1>0$ such that if we put $c(m)=[c_1 m - c_2]_+$ for any integer m, then the following hold true*: 1) *For every $p>0$ and $\varphi \in \Gamma(V, I^m\Omega_X^p)$ with $d\varphi=0$ we can find a $\psi \in \Gamma(V, I^{c(m)}\Omega_X^{p-1})$ such that $\varphi = d\psi$.* 2) *Suppose further that V is contractible. Then for every $p \geqq 0$ and every $\varphi \in \Gamma(V, \Omega_X^p)$ with $d\varphi \in \Gamma(V, I^m\Omega_X^{p+1})$ we can find a $\psi \in \Gamma(V, \Omega_X^{p-1})$ such that $\varphi - d\psi \in \Gamma(V, I^{c(m)}\Omega_X^p)$, where $\Omega_X^{-1}=C_X$ and $d: \Omega_X^{-1} \to O_X$ is the natural inclusion.*

The formal Poincaré analytic lemma mentioned above follows from 2) of the above theorem together with the following:

Remark. For each $y \in Y$ there exists a fundamental system $\{V\}$ of contractible open neighborhoods V of y in X which are good with respect to Y. In fact, since the problem is local, we may assume that $X=C^n := C^n(z_1, \cdots, z_n)$ where $n=\dim X$. Let $r=\sum_{i=1}^n |z_i|^2$ and $D_\varepsilon=\{r<\varepsilon\}$

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