43. On a Conjecture of S. Chowla and of S. Chowla and H. Walum. III

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Let $P_r(v)$ denote the periodic Bernoulli polynomial of degree $r: P_r(v)=B_r(\{v\})$, where $B_r(v)$ is the r-th Bernoulli polynomial, $\{v\}=v-[v]$ being the fractional part of v ($[v]$ is the greatest integer not exceeding v). For $a\in \boldsymbol{R}$ and $r\in \boldsymbol{N}$ we put

$$G_{a,r}(x)=\sum_{n\leqq\sqrt{x}} n^a P_r\left(\frac{x}{n}\right). \tag{1}$$

Then Chowla and Walum's conjecture is that there holds the estimate

$$G_{a,r}(x)=O(x^{a/2+1/4+\varepsilon}) \tag{2}$$

for every positive ε (cf. [3], [6]). The case $r=1$ is concerned with Dirichlet's divisor problem and presents a difficulty of the highest degree, and the case $r=2$ is called Chowla's conjecture [4], [6], which seems to be as deep as the divisor problem itself: For every positive ε and $\psi(v)=\{v\}-\frac{1}{2}$

$$G_{0,2}(x)=\sum_{n\leqq\sqrt{x}}\left\{\psi^2\left(\frac{x}{n}\right)-\frac{1}{12}\right\}=O(x^{1/4+\varepsilon}). \tag{3}$$

We have proved in [6] that a stronger version of (2) is true if $a\geqq\frac{1}{2}$ and $r\geqq 2$, namely we can claim that

$$G_{a,r}(x)=O(x^{a/2+1/4}),\qquad G_{1/2,r}(x)=O(x^{1/2}\log x) \tag{4}$$

in the case specified above, while in case $0\leqq a<\frac{1}{2}$ and $r\geqq 2$ it holds that

$$G_{a,r}(x)=O(x^{(4a+3)/10}). \tag{5}$$

In this note we shall give further developments in the investigation of the conjecture (2) in case $a<\frac{1}{2}$ and $r=2$, namely, we shall state a series representation for $G_{a,2}(x)$ similar to that for $G_{0,2}(x)$ obtained by Wigert [9], an average result for $-\frac{1}{2}<a<\frac{1}{2}$ analogous to that proved by Hardy [5] regarding Dirichlet's divisor problem, and finally

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