40. *Multiple Torsion Theories over Left and Right Perfect Rings*

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(Communicated by Shokichi IYANAGA, M. J. A., April 12, 1980)

Multiple torsion theories were first discussed by Kurata [8]: An *n-fold torsion theory* is an n-tuple $(\boldsymbol{T}_1, \boldsymbol{T}_2, \cdots, \boldsymbol{T}_n)$ of classes of modules such that $(\boldsymbol{T}_i, \boldsymbol{T}_{i+1})$ is always a torsion theory. Kurata showed that there can be only four kinds of n-fold torsion theory. In this note we obtain characterizations of these various types for modules over a left and right perfect ring, the torsion theories being described in terms of properties of the partitions of the simple modules which they induce. Torsion theories over such a ring are closely related to the simple modules: Any TTF class $\boldsymbol{T}$ is both the smallest torsion class and the smallest torsion-free class containing

$$\{S \mid S \text{ is simple and } S \in \boldsymbol{T}\}.$$

The latter result is proved by the dualization of a method we used in [7] to "lift" torsion properties to a ring R from a factor ring R/I where I is right T-nilpotent.

All rings we discuss have identities and all modules are unital left modules. If $\boldsymbol{M}$ is a class of modules over a ring R, we define

$$\boldsymbol{M}^r = \{N \mid \operatorname{Hom}_R(M, N) = 0 \ \forall M \in \boldsymbol{M}\}$$
$$\boldsymbol{M}^l = \{K \mid \operatorname{Hom}_R(K, M) = 0 \ \forall M \in \boldsymbol{M}\}.$$

In most respects we adhere to the usage and conventions of [8].

Let M be a module over a ring R, I an ideal of R. We define submodules $M(\alpha)$ for all ordinals α as follows:

$$M(0)=M; \quad M(\alpha+1)=IM(\alpha); \quad M(\beta)=\bigcap_{\alpha<\beta} M(\alpha) \quad \text{if } \beta \text{ is a limit.}$$

Then for some ordinal μ we have $M(\mu+1)=M(\mu)$. If $M(\mu)=0$, we call

$$M=M(0)\supseteq M(1)\supseteq \cdots \supseteq M(\alpha)\supseteq M(\alpha+1)\supseteq \cdots \supseteq M(\mu)=0 \cdots (*)$$

the *descending I-series* of M.

Proposition 1. *Let M be an R-module with descending I-series* (*), *$\boldsymbol{T}$ a TTF class of R-modules. Then $M \in \boldsymbol{T}$ if and only if $M(\alpha)/M(\alpha+1) \in \boldsymbol{T}$ for each ordinal α.*

Proof. "If": If each $M(\alpha)/M(\alpha+1) \in \boldsymbol{T}$, then $M/M(1)=M(0)/M(1) \in \boldsymbol{T}$. If now $M/M(\alpha) \in \boldsymbol{T}$, it can be seen from the exact sequence

$$0 \to M(\alpha)/M(\alpha+1) \to M/M(\alpha+1) \to M/M(\alpha) \to 0$$

that $M/M(\alpha+1) \in \boldsymbol{T}$. If β is a limit and $M/M(\alpha) \in \boldsymbol{T}$ for all $\alpha<\beta$, we have an embedding