39. *On the General Principal Ideal Theorem*

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By means of the theory of the module of genus, S. Iyanaga and Herbrand established the general principal ideal theorem in [4], [5] and [6]. In this paper, we prove the theorem in an improved form by an investigation of the structure of the idele groups [8].

1. Let k be an algebraic number field, $\mathfrak{m}$ a divisor of k, *which may contain Archimedian primes*, and K the ray class field modulo $\mathfrak{m}$ of k. Denote the conductor, the different and the module of genus of K over k by $\mathfrak{f}_{K/k}$, $\mathfrak{D}_{K/k}$ and $\mathfrak{F}_{K/k}$ respectively. Then as divisors of K, we have

$$\mathfrak{f}_{K/k}=\mathfrak{D}_{K/k}\cdot\mathfrak{F}_{K/k}.$$

For a prime ideal $\mathfrak{P}$ of K, let $e(\mathfrak{P})=e(\mathfrak{P}/k)$ be the order of ramification of $\mathfrak{P}$ over k, i.e.

$$\mathfrak{P}^{e(\mathfrak{P})}\,|\,(\mathfrak{P}\cap k)\cdot O_K, \quad \text{and} \quad \mathfrak{P}^{e(\mathfrak{P})+1}\nmid(\mathfrak{P}\cap k)\cdot O_K,$$

and put

$$\mathfrak{M}_{K/k}=\mathfrak{m}\cdot\mathfrak{D}_{K/k}^{-1}\cdot\prod_{\mathfrak{P}}\mathfrak{P}^{e(\mathfrak{P})-1}=\mathfrak{F}_{K/k}\cdot(\mathfrak{m}\cdot\mathfrak{f}_{K/k}^{-1})\cdot\prod_{\mathfrak{P}}\mathfrak{P}^{e(\mathfrak{P})-1}.$$

Our improved form of the general principal ideal theorem is

Theorem 1. *The extension of an ideal $\mathfrak{a}$ of k into K belongs to the principal ray class modulo $\mathfrak{M}_{K/k}$ if $\mathfrak{a}$ is relatively prime to $\mathfrak{m}$. In other words,*

$$\mathfrak{a}\cdot O_K=A\cdot O_K$$

with $A\in K^\times$ such that

$$A\equiv 1 \qquad \mathrm{mod}\,\mathfrak{M}_{K/k}.$$

Here O_K is the maximal order of K.

In [4] and [6], the general principal ideal theorem was proved for $\mathfrak{F}_{K/k}$ in place of $\mathfrak{M}_{K/k}$ of Theorem 1. Note that $\mathfrak{f}_{K/k}$ divides $\mathfrak{m}$.

2. Let $k_A^\times$ and $K_A^\times$ be the idele groups of k and K respectively, and, $k_\infty^\times$ and $K_\infty^\times$ the Archimedian parts of $k_A^\times$ and $K_A^\times$ respectively. Let $k_{\infty+}^\times$ be the connected component of the unity of $k_\infty^\times$, and $k^\#$ the closure of $k^\times\cdot k_{\infty+}^\times$ in $k_A^\times$. For a prime ideal $\mathfrak{p}$ of k, we denote the $\mathfrak{p}$-adic completion of k by $k_\mathfrak{p}$, the closure of the maximal order O_k in $k_\mathfrak{p}$ by $O_\mathfrak{p}$, and the unit group of $O_\mathfrak{p}$ by $O_\mathfrak{p}^\times$. For an Archimedian prime $\mathfrak{p}_\infty$, the completion of k at $\mathfrak{p}_\infty$ is denoted by $k_{\mathfrak{p}_\infty}$, and the connected component of the unity of $k_{\mathfrak{p}_\infty}^\times$ by $k_{\mathfrak{p}_\infty+}^\times$. For K, for a prime ideal $\mathfrak{P}$ of K, and for an Archimedian prime $\mathfrak{P}_\infty$ of K, we define $K_{\infty+}^\times$, $K^\#$, $K_\mathfrak{P}$, $O_\mathfrak{P}$, $O_\mathfrak{P}^\times$, $K_{\mathfrak{P}_\infty}$ and