37. *Multi-Dimensional Generalizations of the Chebyshev Polynomials. II*[*)]

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4. **Proofs. Proof of Lemma 3.1.** We have

$$\frac{1}{k}P^{-1/2}_{0,m}=\frac{1}{k}P^{-1/2}_{-m,0}=(u_1^{-1})^m+\cdots+(u_{k+1}^{-1})^m.$$

Thus it is necessary to show that

$$r(z)=z^{k+1}-(b^{-1}x_k)z^k+\cdots+(-1)^k(b^{-1}x_1)z+(-1)^{k+1}b^{-1}$$

has roots $u_1^{-1},\cdots,u_{k+1}^{-1}$. This follows from

$$\prod_{i=1}^{k+1}(z-u_i^{-1})=\sum_{i=0}^{k+1}(-1)^i x_i z^{k+1-i}$$

and $\sigma_i(u_1^{-1},\cdots,u_{k+1}^{-1})=b^{-1}\sigma_{k+1-i}(u_1,\cdots,u_{k+1})$. The proof for $P^{1/2}_{-m,0}$ is similar.

Proof of Lemma 3.2. If we allow $b=0$ in the definition of $P^{-1/2}_{m,0}(\underline{x};b)$, then $u_1,\cdots,u_{k+1}$ are the roots of

$$z^{k+1}-x_1z^k+\cdots+(-1)^k x_k z=z(z^k+\cdots+(-1)^k x_k),$$

where $x_i=\sigma_i(\underline{u})$, $\underline{u}=(u_1,\cdots,u_{k+1})$. One of these roots, say u_{k+1}, is zero. Then

$$\frac{1}{k}P^{-1/2}_{m,0}(\underline{x};0)=u_1^m+\cdots+u_k^m$$

where $u_1,\cdots,u_k$ are the roots of

$$z^k-x_1z^{k-1}+\cdots+(-1)^k x_k.$$

This proves the first formula. The proof for $P^{1/2}_{m,0}(\underline{x};0)$ is analogous.

The proof of Lemma 3.4 follows from Definition 2.1 using methods similar to those in [11].

Proof of Theorem 3.5.

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}P^{-1/2}_{m,n}s^mt^n\\
&=\sum_m\sum_n\frac{1}{k^2}P^{-1/2}_{m,0}P^{-1/2}_{-n,0}-\frac{1}{k}P^{-1/2}_{m-n,0}s^mt^n\\
&=\Big(\frac{1}{k}\sum_m P^{-1/2}_{m,0}s^m\Big)\Big(\frac{1}{k}\sum_n P^{-1/2}_{-n,0}t^n\Big)-\frac{1}{k}\sum_m\sum_n P^{-1/2}_{m-n,0}s^mt^n\\
&=\frac{N_+}{D_+}\frac{N_-}{D_-}-\frac{1}{k}\sum_m\sum_n P^{-1/2}_{m-n,0}s^mt^n.
\end{aligned}$$

The last term here can be expressed in terms of D_+, D_-, N_+, N_- by

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