# 36. *Multi-Dimensional Generalizations of the Chebyshev Polynomials. I**)

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**1. Introduction.** This paper continues the study of the classes of polynomials in 2 variables given in Dunn and Lidl [3] and generalizes these polynomials in two ways: They are generalized to polynomials in $k$ variables over an arbitrary field $K$; secondly a parameter $b \in K$ is introduced for these polynomials, similar to the generalization of the classical Chebyshev polynomials in one variable as in Dickson [1] and Schur [14]. In analysis, the most important case, of course, is $K=C$ and $b=1$, which gives a natural generalization of the Chebyshev polynomials, see Koornwinder [8]. However, there are also some interesting algebraic and number theoretic properties in the more general case of a field $K$ and $b \in K$, particularly for $K=GF(q)$ the one-dimensional polynomials have been studied extensively; see Lansch and Nöbauer [9], Fried [6] and Schur [14]. We use the same notation as in [3] and obtain generating functions and recurrence relations for generalized Chebyshev polynomials of the first and second kind in $k$ variables. In the present paper we are not considering any of the analytic properties of the polynomials (for $k=1$ see Rivlin [13] or Szegö [15]), such as partial differential operators or orthogonality. A different approach to give multi-dimensional extensions of Chebyshev polynomials is introduced by Hays [7]. For some properties of special functions in $k$ variables and a bibliography including the earlier papers on the subject we refer to [5]. We have organized the presentation of the material into I and II, each consisting of two sections: § 2 Definitions, § 3 Results in I and § 4 Proofs, § 5 Outlook in II.

**2. Definitions.** Dickson [1] generalized the classical Chebyshev polynomials in the following way. Let $K$ be a field, $r(z)=z^2-xz+b$ a polynomial over $K$ with roots $u$ and $v$ in a suitable extension field $L$ of $K$ (e.g. if $K=C$ then $L=C$, if $K=GF(q)$ then $L=GF(q^2)$). Then generalizations of the Chebyshev polynomials in one variable of the first and second kind are given by (2.1) and (2.2), respectively.

(2.1) $$P_n^{-1/2}(x\,;b)=u^n+v^n, \qquad \text{for } n \in Z$$

(2.2) $$P_n^{1/2}(x\,;b)=(u-v)^{-1}(u^{n+1}-v^{n+1}), \qquad \text{for } n \geq 0,$$

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