33. *Ultradifferentiability of Solutions of Ordinary Differential Equations*

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Let M_p, $p=0,1,2,\cdots$, be a sequence of positive numbers. An infinitely differentiable function f on an open set Ω in $\boldsymbol{R}^n$ is said to be an *ultradifferentiable function of class* $\{M_p\}$ (resp. *of class* (M_p)) if for each compact set K in Ω there are constants h and C (resp. and for each $h>0$ there is a constant C) such that

$$\sup_{x\in K}|D^\alpha f(x)|\leqq Ch^{|\alpha|}M_{|\alpha|}, \qquad |\alpha|=0,1,2,\cdots.$$

We assume that M_p satisfies the following conditions:

$$(1) \qquad M_0=M_1=1;$$

$$(2) \qquad (M_q/q\,!)^{1/(q-1)}\leqq(M_p/p\,!)^{1/(p-1)}, \qquad 2\leqq q\leqq p,$$

and furthermore in case of class (M_p)

$$(3) \qquad \left(\frac{M_p}{p\,!}\right)^2\leqq\left(\frac{M_{p-1}}{(p-1)\,!}\right)\left(\frac{M_{p+1}}{(p+1)\,!}\right), \qquad p=1,2,\cdots,$$

and

$$(4) \qquad M_p/(pM_{p-1})\to\infty \qquad \text{as } p\to\infty.$$

We consider the initial value problem of ordinary differential equation

$$(5) \qquad \begin{cases}\dfrac{dx}{dt}=f(t,x),\\ x(0)=y,\end{cases}$$

where $f(t,x)=(f_1,\cdots,f_n)$ is an n-tuple of functions defined on $(-T,T)\times\Omega$ with a $T>0$ and an open set Ω in $\boldsymbol{R}^n$. We assume the Lipschitz condition in x. Then for each relatively compact open subset Ω_1 of Ω there is a $0<T_1\leqq T$ such that (5) has for each $y\in\Omega_1$ a unique solution $x=x(t,y)$ on the interval $(-T_1,T_1)$.

Our main result is the following

Theorem. *If all components of $f(t,x)$ are ultradifferentiable functions of class $\{M_p\}$ (resp. of class (M_p)) on $(-T,T)\times\Omega$, then the components of the solution $x(t,y)$ are also ultradifferentiable functions of class $\{M_p\}$ (resp. of class (M_p)) on $(-T_1,T_1)\times\Omega_1$.*

Hereafter we denote by $*$ either $\{M_p\}$ or (M_p). The theorem is proved in two steps.

Proposition 1. *If $f(t,x)$ is ultradifferentiable of class $*$ only in x but uniformly in t, then $x(t,y)$ is ultradifferentiable of class $*$ in y uniformly in t.*