16. The Implicit Function Theorem for Ultradifferentiable Mappings

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Let M_p, $p=0,1,2,\cdots$, be a sequence of positive numbers. An infinitely differentiable function f on an open set U in $\boldsymbol{R}^n$ is said to be an *ultradifferentiable function of class* $\{M_p\}$ (*resp. of class* (M_p)) if for each compact set K in U there are constants h and C (resp. and each $h>0$ there is a constant C) such that

$$\sup_{x\in K}|D^\alpha f(x)|\leqq Ch^{|\alpha|}M_{|\alpha|},\qquad |\alpha|=0,1,2,\cdots.$$

A mapping $F=(f_1,\cdots,f_m)$ from an open set U in $\boldsymbol{R}^n$ into $\boldsymbol{R}^m$ is said to be *ultradifferentiable of class* $\{M_p\}$ (resp. (M_p)) if all components f_i are ultradifferentiable functions of class $\{M_p\}$ (resp. (M_p)).

We assume that M_p satisfies the following conditions:

$$M_0=M_1=1; \tag{1}$$

There is a constant H such that

$$(M_q/q!)^{1/(q-1)}\leqq H(M_p/p!)^{1/(p-1)},\qquad 2\leqq q\leqq p; \tag{2}$$

Furthermore in case of class (M_p)

$$\frac{M_p}{pM_{p-1}}\to\infty\qquad \text{as}\quad p\to\infty. \tag{3}$$

Then we have

The inverse mapping theorem. *If $F=(f_1,\cdots,f_n)$ is an ultradifferentiable mapping of class $\{M_p\}$ (resp. (M_p)) from an open set U in $\boldsymbol{R}^n$ into an open set V in $\boldsymbol{R}^n$ and if the Jacobian*

$$\frac{\partial(f_1,\cdots,f_n)}{\partial(x_1,\cdots,x_n)}=\det\left(\frac{\partial f_i}{\partial x_j}\right)$$

does not vanish at x^0 in U, then there exist an open neighborhood U_0 of x^0 in U and an open neighborhood V_0 of $y^0=F(x^0)$ in V such that F restricted to U_0 is a homeomorphism onto V_0 and the inverse on V_0 is an ultradifferentiable mapping of class $\{M_p\}$ (resp. (M_p)).

Proof. By the inverse mapping theorem for C^∞ mappings there are open neighborhoods U_0 and V_0 such that $F: U_0\to V_0$ is a C^∞ diffeomorphism. We may assume that the inverse matrix of $(\partial f_i/\partial x_j)$ is uniformly bounded on U_0. To estimate the derivatives of the inverse mapping $F^{-1}=(g_1,\cdots,g_n): V_0\to U_0$, we assume that 0 is an arbitrary point in U_0 and F maps it to 0 in V_0.

Let (a_{ij}) be the inverse matrix of $(\partial f_i/\partial x_j)$ at 0 in U_0. We set