90. A Class of General Boundary Conditions for Multi-Dimensional Diffusion Equation

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1. Let D be the upper half space $R^N_+=\{(x_1, \cdots, x_N)\in R^N | x_N>0\}$ of R^N, or a bounded open domain with smooth boundary in R^N, and let

$$(1)\qquad \frac{\partial u}{\partial t}=Au=\sum_{1\le i, j\le N} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(t, x) + \sum_{1\le i\le N} b_i(x)\frac{\partial u}{\partial x_i}(t, x)+c(x)u(t, x)$$

be a diffusion equation on D with real smooth coefficients defined on $\bar{D}=D\cup\partial D$.

Here, we should like to introduce an existence theorem for (1) with boundary conditions of type

$$(2)\qquad Lu(x)=\tilde{A}u(x)+\delta(x)Au(x)+\partial u/\partial n(x)+\nu[u](x)=0,$$

where $\tilde{A}$ is an elliptic differential operator with real coefficients on ∂D

$$(3)\qquad \tilde{A}u(x)=\sum_{0\le|\alpha|\le 2n}\tilde{a}_\alpha(x)\tilde{D}^\alpha u(x), \qquad x\in\partial D,$$

$\tilde{D}^\alpha u(x)=\partial^{|\alpha|}u(x)/\partial\xi^{\alpha_1}_{1,x}\cdots\partial\xi^{\alpha_{N-1}}_{N-1,x}$, $\alpha=(\alpha_1, \cdots, \alpha_{N-1})$. $\{\xi_{i,x}(y), 1\le i\le N\}$ is a local coordinate near $x\in\partial D$, and is also a set of bounded functions on a neighbourhood of $\bar{D}$. $\delta(x)$ is a non-positive function on ∂D. $u\to\nu[u]$ is an integro-differential operator of type

$$(4)\qquad \nu[u](x)=(-1)^{[m/2]}\int_{\bar{D}\setminus\{x\}}\Big(u(y)-\sum_{0\le|\alpha|\le m}\frac{1}{\alpha!}\tilde{D}^\alpha u(x)\xi^\alpha_x(y)\Big)\nu(x, dy),$$

where $\xi^\alpha_x(y)=\xi^{\alpha_1}_{1,x}(y)\cdots\xi^{\alpha_{N-1}}_{N-1,x}(y)$ and $\alpha!=\alpha_1!\cdots\alpha_{N-1}!$. $\nu(x, \cdot)$ is a measure on $\bar{D}\setminus\{x\}$ such that, for each neighbourhood U_x of $x\in\partial D$,

$$(5)\qquad \int_{U_x\setminus\{x\}}\Big(\sum_{1\le i\le N-1}|\xi_{i,x}(y)|^{m+1}+\xi_{N,x}(y)\Big)\nu(x, dy)+\nu(x, \bar{D}\setminus U_x)<\infty.$$

$\dfrac{\partial}{\partial n}$ is the inward directed normal derivative defined relative to $\{a_{ij}(x)\}$. The detailed proof of our existence theorem will be published elsewhere.

In case $m=n=1$, (2) was obtained by Wentzell [1] as a necessary condition for positive solutions of (1) on a certain set up. The sufficiency was proved by [1], Ueno [2] or Sato-Ueno [3], Bony *et al.* [4], Taira [5], Ueno [6] or [7], and others under auxiliary conditions.

The results for general m and n in this paper were motivated by the method in [7], where (*conditional*) *positive definiteness* is essential instead of the *positivity* in the case of $m=n=1$. Another motivation