75. *Studies on Holonomic Quantum Fields. XVI*

Density Matrix of Impenetrable Bose Gas

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In this article we report the following result concerning a system of impenetrable bosons in one dimension at zero temperature: The one particle reduced density matrix $\rho(x)$ satisfies a non-linear differential equation, an equivalent of a Painlevé equation of the fifth kind. This enables us to calculate the small and large x behaviors of $\rho(x)$ to an arbitrary order.

For the statement of the problem see [1] and references cited therein. As mentioned in [2], our calculation is done by relating the problem to the preceding result obtained there concerning the double scaling limit of the XY model.

Main results are summarized in § 1. Their derivation is briefly described in §§ 2–3.

§ 1. Results. Let $\rho(|x-x'|)$ denote the thermodynamic limit of the one particle reduced density matrix with $\rho_0=\rho(0)$ normalized to be π^{-1} (for the definition see [1]). It is known ([3]) that $\rho(x)$ is an entire function of x.

We find that $\rho(x)$ is expressed as

$$(1)\qquad \rho(x)=\rho_0 \exp\int_0^x dx'\left(\frac{x'}{4y(1-y)^2}\left(\left(\frac{dy}{dx'}\right)^2+4y^2\right)-\frac{(1+y)^2}{4x'y}\right)\quad \text{with } y=y(x'),$$

where $y=y(x)$ is a solution of the following Painlevé equation of the fifth kind:

$$(2)\qquad \frac{d^2y}{dx^2}=\left(\frac{1}{2y}+\frac{1}{y-1}\right)\left(\frac{dy}{dx}\right)^2-\frac{1}{x}\frac{dy}{dx}+\frac{(y-1)^2}{x^2}\left(\alpha y+\frac{\beta}{y}\right)+\frac{\gamma y}{x}+\frac{\delta y(y+1)}{y-1}$$

$$\text{with } \alpha=\frac{1}{2},\ \beta=-\frac{1}{2},\ \gamma=-2i,\ \delta=2.$$

If we set

$$(3)\qquad \sigma(x)=x\frac{d}{dx}\log\rho(x),$$

then $\sigma=\sigma(x)$ itself satisfies the non-linear ordinary differential equation

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