69. On Sufficient Conditions for the Boundedness of Pseudo-Differential Operators

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We report here that pseudo-differential operators are bounded in L_p, $1<p<\infty$, if some considerably weak conditions on the smoothness of their symbols are satisfied.

1. Notations. If $x=(x_1, \cdots, x_n)$ is a point in the n-dimensional Euclidean space $\boldsymbol{R}^n$, and $\alpha=(\alpha_1, \cdots, \alpha_n)$ a multi-index, then we write $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, $\partial_x^\alpha=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}$, $\partial_{x_j}=\partial/\partial x_j$, $|x|=(x_1^2+\cdots+x_n^2)^{1/2}$, $\langle x\rangle=(1+|x|^2)^{1/2}$, $|\alpha|=\alpha_1+\cdots+\alpha_n$. We denote by Δ the difference operator, and adopt the following conventions:

$$\Delta_y a(x,\xi,x')=a(x+y,\xi,x')-a(x,\xi,x'),$$
$$\Delta_\eta a(x,\xi,x')=a(x,\xi+\eta,x')-a(x,\xi,x'),$$
$$\Delta_{y'} a(x,\xi,x')=a(x,\xi,x'+y')-a(x,\xi,x').$$

Let $a(x,\xi,x')$ be a symbol, that is, a continuous function of (x,ξ,x') in $\boldsymbol{R}^{3n}$. If m is a non-negative integer, and $0<\theta<1$, we define

$$\|a\|_m=\sup_{x,\xi,x',|\alpha|\leqq m}|\partial_\xi^\alpha a(x,\xi,x')|\langle\xi\rangle^{|\alpha|},$$
$$|a|_{m+\theta}=\sup_{x,\xi,x',|\eta|\leqq\langle\xi\rangle/2,|\alpha|=m}|\Delta_\eta\partial_\xi^\alpha a(x,\xi,x')|\langle\xi\rangle^{m+\theta}|\eta|^{-\theta},$$
$$\|a\|_{m+\theta}=\|a\|_m+|a|_{m+\theta}.$$

If t and σ are positive numbers, we define

$$\omega_\sigma(a;t)=\sup_{|y|\leqq t}\|\Delta_y a(x,\xi,x')\|_\sigma,$$
$$\omega'_\sigma(a;t)=\sup_{|y'|\leqq t}\|\Delta_{y'} a(x,\xi,x')\|_\sigma.$$

It is easy to find that $\|a\|_\sigma\leqq c\|a\|_\tau$, $\omega_\sigma(a;t)\leqq c\omega_\tau(a;t)$, $\omega'_\sigma(a;t)\leqq c\omega'_\tau(a,t)$ if $\sigma<\tau$, where c is a constant independent of a and t.

2. Main results. Our main results are stated as follows:

Theorem 1. *If a symbol $a(x,\xi)$ satisfies the conditions*

(a) $\|a\|_\sigma$ *is finite, and*

(b) $\omega_\sigma(a;t)\in L_2^*$ $(=L_2([0,1], t^{-1}dt))$

for some $\sigma>n/2$, then the pseudo-differential operator $a(X,D)$ is bounded in $L_2(\boldsymbol{R}^n)$.

If a symbol $a(\xi,x')$ satisfies the conditions (a) *and*

(b′) $\omega'_\sigma(a;t)\in L_2^*$

for some $\sigma>n/2$, then the pseudo-differential operator $a(D_X,X')$ is

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