50. *Invariants of Reflection Groups in Positive Characteristics*

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Let k be a field of characteristic $p>0$. Let V be a vector space over the field k and $k[V]$ be the symmetric algebra of V. Let G be a finite subgroup of $GL(V)$ with $p\,|\,|G|$. Then G can be regarded as a subgroup of the automorphism group of $k[V]$. In [5], we classified irreducible groups G such that the invariant subrings $k[V]^G$ are polynomial rings under certain conditions. For a reflection group G, it is well known (e.g. [1]) that $k[V]^G$ is a unique factorization domain, but it has not been known whether $k[V]^G$ is a Macaulay ring. In this note we give some examples of reflection groups G such that $k[V]^G$ are not Macaulay rings.

Suppose that $p>2$ and that n is an integer with $p\,|\,n$, $n\geqq 5$. Let $E=\oplus_{i=1}^n kT_i$, $V=\oplus_{i=2}^n k(T_i-T_1)$ and $V'=V/k\sum_{i=1}^n T_i$ be vector spaces over k. The symmetric group S_n acts on $\{T_1,\cdots,T_n\}$ as permutations. Then the k-spaces V and V' are naturally regarded as S_n-faithful kS_n-modules. The group S_n is generated by reflections in $GL(V)$ and $GL(V')$ respectively. It is proved in [5] that $k[V]^{S_n}$ and $k[V']^{S_n}$ are not polynomial rings. The purpose of this note is to show the following stronger result:

Theorem. *Suppose that* $p\,\|\,n$ *and* $p\geqq 7$. *Then* $k[V]^{S_n}$ *and* $k[V']^{S_n}$ *are not Macaulay rings.*

Proof. Set $X_i=T_i-T_1$ $(2\leqq i\leqq n)$ and

$$u=\begin{bmatrix}1&1&1\cdots 1\\1&2&1\cdots 1\\1&1&2\cdots 1\\&\cdots\cdots&\\1&1&1\cdots 2\end{bmatrix}\in GL_{n-1}(k).$$

Put ${}^t[Y_2,\cdots,Y_n]=u\,{}^t[X_2,\cdots,X_n]$ and denote by $\overline{Y}_i$ $(3\leqq i\leqq n)$ the canonical images of Y_i in V'. Let

$$a={}^t[1,2,\cdots,p-1,0,1,\cdots,p-1,\cdots,0,1,\cdots,p-1]\in k^{n-1}$$

and choose the element $a'={}^t[a_3',\cdots,a_n']\in k^{n-2}$ such that $ua=\begin{bmatrix}0\\a'\end{bmatrix}$. Put

$W_2=X_2-1,\ W_3=X_3-2,\ \cdots,\ W_p=X_p-(p-1),\ W_{p+1}=X_{p+1},$
$W_{p+2}=X_{p+2}-1,\ \cdots,\ W_{2p}=X_{2p}-(p-1),\ \cdots,\ W_{n-p+1}=X_{n-p+1},$
$W_{n-p+2}=X_{n-p+2}-1,\ \cdots,\ W_n=X_n-(p-1).$

Let M be the maximal ideal of $k[V]$ generated by the set $\{W_2,\cdots,W_n\}$