41. A Remark on the Hadamard Variational Formula

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§1. **Introduction.** Let $f(x)$ be a real valued C^∞ function of x in $\boldsymbol{R}^2$. Using this and real number $t \in \boldsymbol{R}$, we define the open set $\Omega_t = \{x \in \boldsymbol{R}^2 \mid f(x) < t\}$. Its boundary is $\gamma_t = \{x \in \boldsymbol{R}^2 \mid f(x) = t\}$. We assume the following assumptions for $f(x)$;

(A. 1) *Ω_1 is a non empty simply connected bounded domain in $\boldsymbol{R}^2$.*

(A. 2) *All the $t \in [-1, 0) \cup (0, 1]$ are regular values of f.*

(A. 3) *Ω_1 contains only one critical point x^0 of f. At this point, the function $f(x^0)$ has its value* 0 *and it has non-degenerate Hessian of signature of type* (1, 1).

We shall consider the Green function $g_t(x, y)$ for the Dirichlet problem in the open set Ω_t for any $t \in [-1, 1]$, that is, $g_t(x, y)$ is the solution for the following boundary value problem;

$$(1) \qquad -\Delta g_t(x, y) = \delta(x-y) \qquad \text{for any } x, y \text{ in } \Omega_t.$$

and

$$(2) \qquad g_t(x, y) = 0, \qquad \text{if } x \in \gamma_t,\ y \in \Omega_t.$$

When t decreases from 1 to any $\varepsilon > 0$, the open set Ω_t shrinks to $\Omega\varepsilon$. Throughout this process Ω_t is a simply connected domain with its smooth boundary, because (A. 2) and (A. 3) hold. See, for example Milnor [6]. Therefore, the celebrated Hadamard variational formula implies that $(d/dt)g_t(x, y)$ exists for $t \neq 0$ and for any x and y in Ω_t and that

$$(3) \qquad \frac{d}{dt} g_t(x, y) = \int_{\gamma_t} \frac{\partial g_t(x, z)}{\partial \nu_z} \frac{\partial g_t(y, z)}{\partial \nu_z} \frac{1}{|\operatorname{grad} f(z)|} d\sigma_z,$$

where $d\sigma_z$ is the line element of γ_t and ν_z is the unit outer normal to γ_t at z. (See Hadamard [5], Garabedian [4], Garabedian-Schiffer [3]. Simpler proof is given in Fujiwara-Ozawa [2].) This enables us to write

$$(4) \qquad g_1(x, y) - g_\varepsilon(x, y) = \int_\varepsilon^1 \frac{d}{dt} g_t(x, y) dt$$

for any $x \neq y$ in Ω_ε if $\varepsilon > 0$. Hence the following natural question arises.

(Q) *Can one replace ε in* (4) *by* -1?

This does not seem a trivial problem because the open set Ω_t has two connected components for $t \leq 0$ while it is connected for $t > 0$. The aim of this note is to prove the following affirmative answer to this question (Q).

Theorem 1. *For any $x \neq y$ in Ω_{-1}, we have*