25. On Zariski Problem

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In this note we generalize a result of Zariski [8, § 7]. As an application, using the theory of Miyanishi [5], [6], we prove the following

Theorem. *Let S be a surface defined over a field k of characteristic zero such that $S\times A^1\cong A^3$. Then $S\cong A^2$.*

Namely the so-called Zariski problem is solved in the affirmative way. Our method of proof will work also in positive characteristic cases provided that there is a sufficiently powerful analogue of the theory of Iitaka [1], [2]. It should be emphasized that the theory of Miyanishi plays a very important role in our proof.

§ 1. Zariski decomposition of pseudo effective line bundles. Let S be a complete non-singular surface defined over an algebraically closed field k of any characteristic. *Prime divisor* means an irreducible reduced curve on S.

(1.1) A linear combination of prime divisors with coefficients in the rational number field $\boldsymbol{Q}$ is called a *$\boldsymbol{Q}$-divisor*. A $\boldsymbol{Q}$-divisor is said to be *effective* if each coefficient is non-negative.

(1.2) An element of $Pic(S)\otimes\boldsymbol{Q}$ is called a *$\boldsymbol{Q}$-line bundle*. Any $\boldsymbol{Q}$-divisor D defines naturally a $\boldsymbol{Q}$-line bundle, which is denoted by D by abuse of notation. For any $\boldsymbol{Q}$-line bundles F_1 and F_2, we define the intersection number $F_1F_2\in\boldsymbol{Q}$ in the obvious way.

(1.3) A $\boldsymbol{Q}$-line bundle H is said to be *semi-positive* if $HC\geqq 0$ for any prime divisor C. Then, obviously, $HE\geqq 0$ for any effective $\boldsymbol{Q}$-divisor E.

(1.4) **Lemma.** *Let H be a semipositive $\boldsymbol{Q}$-line bundle and let E be an effective $\boldsymbol{Q}$-divisor. If $(H+E)C_i\geqq 0$ for each prime component C_i of E, then $(H+E)$ is semipositive.*

Proof is easy.

(1.5) A $\boldsymbol{Q}$-line bundle L is said to be *pseudo effective* if $LH\geqq 0$ for any semipositive $\boldsymbol{Q}$-line bundle H. Clearly any effective $\boldsymbol{Q}$-divisor is pseudo effective.

(1.6) Let $C_1,\cdots,C_q$ be prime divisors. By $V(C_1,\cdots,C_q)$ we denote the $\boldsymbol{Q}$-vector space of $\boldsymbol{Q}$-divisors generated by $C_1,\cdots,C_q$. $I(C_1,\cdots,C_q)$ denotes the quadratic form on $V(C_1,\cdots,C_q)$ defined by the self intersection number.