72. *Parallel Vector Fields and the Betti Number*

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(Communicated by Kunihiko KODAIRA, M. J. A., Nov., 13, 1978)

Introduction. Let M^n be an n dimensional connected compact orientable smooth Riemannian manifold. In the previous paper [3] we showed that the Betti numbers of M^n with one or two parallel vector fields satisfy some inequalities. In this note we shall generalize these results to the case of M^n admitting r parallel vector fields ($1 \leqq r \leqq n$). A trivial example of such M^n is the Riemannian product $T^r \times M^{n-r}$, where T^r is the flat r-torus and M^{n-r} is any Riemannian manifold.

1. Preliminaries. Let $\mathcal{H}_p$ be the vector space of harmonic p-forms on M^n. $\dim \mathcal{H}_p$ is equal to the p-th Betti number b_p. We make a convention that $\mathcal{H}_p = \{0\}$ for $p > n$ or $p < 0$ and hence all operators act trivially on such spaces. Throughout the paper we shall denote by p any integer.

Let u be a vector field on M^n. By the natural identification with respect to the Riemannian metric, u is identified with a 1-form which will be denoted by u again. $e(u)$ and $i(u)$ denote respectively the operators of exterior and interior product by u. For a p-form ω, we have $e(u)\omega = u \wedge \omega$ and

$$(i(u)\omega)(X_1, \cdots, X_{p-1}) = \omega(u, X_1, \cdots, X_{p-1})$$

where $X_1, \cdots, X_{p-1}$ are tangent vectors. These operators satisfy $e(u)^2 = i(u)^2 = 0$. $i(u)$ is an anti-derivation and hence

$$i(u)e(u) + e(u)i(u) = I \tag{1}$$

holds for a unit vector field u, where I is the identity on p-form.

2. Parallel vector fields. Let u be a parallel vector field on M^n. First we notice that $\omega \in \mathcal{H}_p$ implies $e(u)\omega \in \mathcal{H}_{p+1}$ and $i(u)\omega \in \mathcal{H}_{p-1}$.

Now we assume that M^n admits r ($1 \leqq r \leqq n$) linearly independent parallel vector fields $u_1, \cdots, u_r$. Making use of the Schmidt process, we may suppose that $u_1, \cdots, u_r$ are orthonormal, i.e.,

$$i(u_k)u_j = \delta_{kj} \qquad (1 \leqq k,\ j \leqq r).$$

$a_1, \cdots, a_k$ ($1 \leqq a_1, \cdots, a_k \leqq r$) being integers, let us define

$$i_{a_1 \cdots a_k} = i(u_{a_1}) \cdots i(u_{a_k}) : \mathcal{H}_p \to \mathcal{H}_{p-k},$$
$$e_{a_1 \cdots a_k} = e(u_{a_1}) \cdots e(u_{a_k}) : \mathcal{H}_p \to \mathcal{H}_{p+k}.$$

Lemma. *For* $1 \leqq s \leqq r$, *we have*

$$I = -\sum_{k=1}^{s} \sum_{1 \leqq a_1 < \cdots < a_k \leqq s} (-1)^{k(k+1)/2} e_{a_1 \cdots a_k} i_{a_1 \cdots a_k} + (-1)^{s(s-1)/2} i_{1 \cdots s} e_{1 \cdots s}. \tag{2}$$