70. *Remarks on the Existence of Finite Invariant Measures for Groups of Measurable Transformations*

By Kôkichi SAKAI
Department of Mathematics, Kagoshima University

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0. Introduction. Throughout this note let $(X, \mathfrak{B}, m)$ be a finite measure space and let G be an infinite group of invertible bi-measurable non-singular transformations of X onto itself. A measure μ on $(X, \mathfrak{B})$ is called *G-invariant* if $\mu(gE)=\mu(E)$ for all $g \in G$ and $E \in \mathfrak{B}$. By A. Hajian and Y. Ito [1] it is proved that there exists a finite G-invariant measure on $(X, \mathfrak{B})$ equivalent to m if and only if in $\mathfrak{B}$ there does not exist any weakly G-wandering set of positive m-measure. Making use of the elegant result, in this note, we shall give some necessary and sufficient conditions for the existence of a finite G-invariant measure on $(X, \mathfrak{B})$ equivalent to m. Our results have been shown by Hopf [3], Kubokawa [4], Hajian and Kakutani [2] for the case when G is a cyclic group.

1. The main theorem. To state our results, we begin with some definitions. By N we denote the set of all positive integers. In what follows let A, B, A_i, B_i $(i \in N)$ and W be subsets of X in $\mathfrak{B}$.

Definition 1. *A is equivalent to B under G, denoted by $A \sim B$, if A and B can be expressed as countable disjoint union $A=\bigcup_{i=1}^{\infty} A_i$ and $B=\bigcup_{i=1}^{\infty} B_i$ such that there exists a sequence $\{g_i ; i \in N\}$ in G satisfying $g_i A_i=B_i$ for all $i \in N$.*

Definition 2. *A is G-bounded if $m(A-B)=0$ for any $B \subset A$ with $B \sim A$.*

Definition 3. *$(X, \mathfrak{B}, m)$ is G-compact if for any $\varepsilon>0$ there corresponds a $\delta>0$ such that $m(A)<\delta$ and $B \sim A$ imply $m(B)<\varepsilon$.*

Definition 4. *W is weakly G-wandering if there exists a sequence $\{g_i ; i \in N\}$ in G such that $g_i W \cap g_j W=\emptyset$ for all $i, j \in N$ with $i \neq j$.*

Definition 5. *A family Λ of measures on $(X, \mathfrak{B})$ is equi-uniformly absolutely continuous with respect to m if for any $\varepsilon>0$ there corresponds a $\delta>0$ such that $m(B)<\delta$ implies $\lambda(B)<\varepsilon$ for all $\lambda \in \Lambda$.*

For every $g \in G$ let m_g be the measure on $(X, \mathfrak{B})$ defined by $m_g(E)=m(gE)$ for any $E \in \mathfrak{B}$. Now we consider the following conditions:

(0) *There exists a finite G-invariant measure μ on $(X, \mathfrak{B})$ which is equivalent to m.*

(1) *$(X, \mathfrak{B}, m)$ is G-compact.*

(2) *X is G-bounded.*