58. *Studies on Holonomic Quantum Fields. V*

By Mikio SATO, Tetsuji MIWA, and Michio JIMBO
Research Institute for Mathematical Sciences, Kyoto University

(Communicated by Kôsaku YOSIDA, M. J. A., Dec. 12, 1977)

This is a continuation of the series of our notes [1].

Here we shall give a summary of the theory of Clifford group. As for details see [2]. We remark that we have changed the definition of T_g and nr (g) which was given in [1].

1. Norms and rotations. Let W be an N dimensional vector space over $\boldsymbol{C}$. We set $W^*=\mathrm{Hom}_{\boldsymbol{C}}(W,\boldsymbol{C})=\{\eta\,|\,\eta: W\to\boldsymbol{C}, w\mapsto\eta(w)\}$. Let $\Lambda(W)=\bigoplus_{\mu=0}^{N}\Lambda^{\mu}(W)$ denote the exterior algebra over W. We denote by δ the linear homomorphism $\delta: W^*\to\mathrm{End}_{\boldsymbol{C}}(\Lambda(W))$, $\eta\mapsto\delta_\eta$ which satisfies $\delta_\eta(1)=0$ and $\delta_\eta(wa)=\eta(w)a-w\delta_\eta(a)$ for $w\in W$ and $a\in\Lambda(W)$. Let κ be an element of $\mathrm{Hom}_{\boldsymbol{C}}(W,W^*)$ such that $\iota\underset{\mathrm{def}}{=}\kappa+{}^t\kappa$ is invertible. An orthogonal structure is introduced to W by the inner product $\langle w,w'\rangle=\iota(w)(w')=\iota(w')(w)$. We denote by $A(W)$ the Clifford algebra over the orthogonal space W thus obtained.

There exists a unique linear isomorphism

$$\mathrm{Nr}_\kappa: A(W)\to\Lambda(W),\qquad a\mapsto\mathrm{Nr}_\kappa(a) \tag{1.1}$$

which satisfies $\mathrm{Nr}_\kappa(1)=1$ and

$$\mathrm{Nr}_\kappa(wa)=w\,\mathrm{Nr}_\kappa(a)+\delta_{\kappa(w)}(\mathrm{Nr}_\kappa(a)). \tag{1.2}$$

We call $\mathrm{Nr}_\kappa(a)$ the κ-norm of a. The constant term of $\mathrm{Nr}_\kappa(a)$ is called the κ-expectation value and is denoted by $\langle a\rangle_\kappa$.

There exists a unique automorphism $a\mapsto\varepsilon(a)$ (resp. anti-automorphism $a\mapsto a^*$) of $A(W)$ characterized by $\varepsilon(w)=-w$ (resp. $w^*=w$) for $w\in W$. We denote by $G(W)$ the Clifford group $\{g\in A(W)\,|\,{}^{\exists}g^{-1}\in A(W), gW\varepsilon(g)^{-1}=W\}$. We denote by T the group homomorphism $T: G(W)\to O(W)$, $g\mapsto T_g$ defined by $T_g(w)=gw\varepsilon(g)^{-1}$ for $w\in W$. Then we have the following exact sequence.

$$1\longrightarrow\mathrm{GL}(1,\boldsymbol{C})\xrightarrow{\mathrm{id.}}G(W)\xrightarrow{T}O(W)\longrightarrow 1. \tag{1.3}$$

A group homomorphism $\mathrm{nr}: G(W)\to\mathrm{GL}(1,\boldsymbol{C})$, $g\mapsto\mathrm{nr}(g)$ is defined by $\mathrm{nr}(g)=g\varepsilon(g)^*$, which is called the spinorial norm of g.

In what follows we shall adopt the following identifications: $\mathrm{Hom}_{\boldsymbol{C}}(W_1\otimes_{\boldsymbol{C}}W_2,\boldsymbol{C})\cong W_2^*\otimes_{\boldsymbol{C}}W_1^*\cong\mathrm{Hom}_{\boldsymbol{C}}(W_1,W_2^*)$.

If $g\in G(W)$, we have

$$\langle g\rangle_\kappa^2=\mathrm{nr}(g)\det((\kappa T_g+{}^t\kappa)\iota^{-1}). \tag{1.4}$$

If $\langle g\rangle_\kappa\neq 0$, we have

$$\mathrm{Nr}_\kappa(g)=\langle g\rangle_\kappa\exp(\rho_g/2) \tag{1.5}$$