83. On a Theorem of F. L. Spitzer and C. J. Stone

By Takesi WATANABE

Department of Applied Science, Kyushu University, Fukuoka, Japan

(Comm. by Z. SUETUNA, M.J.A., July 12, 1961)

1. **Introduction.** In their recent work [6], Spitzer and Stone have proved the following interesting theorem which was the basis of their discussion. Consider a sequence $\{c_k; k=0, \pm 1, \pm 2, \cdots\}$ satisfying the conditions:

(c.1) $$c_k \geqq 0, \quad \sum_{k=-\infty}^{\infty} c_k = 1,$$

(c.2) $$0 < \sum_{k=-\infty}^{\infty} k^2 c_k = v < +\infty,$$

(c.3) $$c_k = c_{-k},$$

(c.4) $$\text{g.c.d. } \{k; k>0, c_k>0\} = 1.$$

Putting $\varphi(\theta)=\sum_{k=-\infty}^{\infty} c_k e^{ik\theta}$ and noting $2 \geqq 1-\varphi(\theta) \geqq 0$, it follows that there exists a unique sequence of polynomials $\{p_n(z)=\sum_{k=0}^{n} p_{nk} z^k;\ p_{nn}>0,\ n=0, 1, 2, \cdots\}$ satisfying

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} p_n(e^{i\theta})\overline{p_m(e^{i\theta})}[1-\varphi(\theta)]d\theta = \delta_{nm}$$

for $n, m=0, 1, 2, \cdots$.

THEOREM (Spitzer and Stone). *The relation*

$$p_{nk} - (2/v)^{\frac{1}{2}}(k/n) \to 0 \quad (n-k\to\infty)$$

holds uniformly in k and n.

In this note we shall derive a more probabilistic version of the above theorem under a weaker condition (c.3)$'$ $\sum_{k=-\infty}^{\infty} kc_k=0$ instead of (c.3). The main feature of our discussion is in full use of the general theory of Markov chains. By doing so we can prove Theorem 2.1 in [6] under (c.3)$'$ and substitute some simple probabilistic arguments for the rather complicated calculations in [6] (e.g. Lemmas 5–11).

2. **Markov chains.** We now summarize some fundamental facts on Markov chains (with discrete parameter). As to the details, we refer the reader to Chap. I of [7].

Let S be a finite or denumerable space and $T=(T(x,y); x, y\in S)$, a *substochastic matrix*[1] on S. Adding a new point e (called 'extra' point) to S, we extend T to $\tilde{S}=S\cup\{e\}$ as follows: $T(x,e)=1-\sum_{y\in S} T(x,y)$, $T(e,e)=1$ and $T(e,y)=0$ for $y\in S$. For any x in $\tilde{S}$, the new transition

1) $\sum_{y\in S} T(x,y)\leqq 1$ for every $x\in S$.