81. On the Bend of Continuous Plane Curves

By Kanesiroo ISEKI
Department of Mathematics, Ochanomizu University, Tokyo
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In the theory of functions of a real variable there is a beautiful theorem of importance due to S. Banach (cf. Saks [2], p. 280):

THEOREM OF BANACH. *Let $F(x)$ be a continuous real function on a linear closed interval I and let $s(y)$ denote for each real number y the (finite or infinite) number of the points of I at which F assumes the value y. Then the function $s(y)$ is B-measurable and its integral over the real line coincides with* $\mathrm{W}(F; I)$, *i.e. the absolute variation of F over I.*

The condition that I is a *closed* interval is not essential for the validity of the assertion. With slight modifications in the proof we have the same result even when I is an arbitrary interval of real numbers; only we then interpret $\mathrm{W}(F; I)$ as the weak variation of F over I (defind on p. 221 of Saks [2]).

We established in our paper [1] certain basic properties of a geometric quantity called curve bend. It is the object of the present note to obtain an analogue of the Banach theorem for the bend of a plane curve determined by an equation of the form $y=F(x)$, where again continuity is the sole condition that we impose upon the function F. Though our theorem is similar to that of Banach in enunciation, the proof turns out far more complicated in our case. We presuppose complete knowledge of [1] on the part of the reader. The precise statement of our theorem reads as follows:

THEOREM. *Let us define $p(\theta)=\langle\cos\theta, \sin\theta\rangle$ for the points θ of the interval $K=[-\pi/2, \pi/2]$. Given on a linear interval I_0 (of any type) a continuous real function $F(x)$, let $f(\theta)$ denote for each $\theta\in K$ the number (finite or infinite) of the points of I_0 at which the unit-vector $p(\theta)$ is a derived direction* (see [1] §42) *for the curve φ defined on I_0 by $\varphi(x)=\langle x, F(x)\rangle$. Then $f(\theta)$ is a B-measurable function on K and its integral over K coincides with $\Omega(\varphi)$, i.e. the bend of φ.*

All the notations of this theorem will be retained throughout the rest of the present note. Since the function $p(\theta)$ is continuous and biunique, so is also its inverse function p^{-1}, which maps the semicircle $p[K]$ onto the interval K. It is immediately seen further that if θ_1 and θ_2 are any pair of points of K, then the angle $p(\theta_1)\diamond p(\theta_2)$ is equal to $|\theta_1-\theta_2|$ (see [1] §21).

This being so, let us begin by proving the following analogue of