57. A note on generalized convex functions.

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§ 1. We are concerned with real finite functions $f(x)$ defined on a closed interval $a \leqq x \leqq b$. E. F. Beckenbach[1] has given a generalization of the notion of a convex function as follows.

Let $F(x; \alpha, \beta)$ be a two-parameter family of real finite functions defined on $a \leqq x \leqq b$ and satisfying the follwing conditions:

(1) each $F(x; \alpha, \beta)$ is a continuous function of x;

(2) there is a unique member of the family which, at arbitrary x_1, x_2 satisfying $a \leqq x_1 < x_2 \leqq b$, takes on arbitrary values y_1, y_2.

The members of the family $F(x; \alpha, \beta)$ are denoted simply by $F(x)$, individual members being distinguished by subscripts. In particular, $F_{ij}(x)$ denotes the member satisfying $F_{ij}(x_i) = f(x_i)$, $F_{ij}(x_j) = f(x_j)$, $(a \leqq x_i < x_j \leqq b)$.

We call *a function* $f(x)$ *to be convex in Beckenbach's sense if*

$$f(x) \leqq F_{12}(x)$$

for all x_1, x_2, x, with $a \leqq x_1 < x < x_2 \leqq b$.

Now let the family $F(x; \alpha, \beta)$ satisfy the following condition (3) in addition to (1) and (2):

(3) let $F(x)$, $F'(x)$ be the members of the family passing through arbitrary points (x_1, y_1), (x_2, y_2); (x_1, y_1'), (x_2, y_2') respectively, then, the member $F_\lambda(x) (\lambda > 0)$ which passes through $(x_1, \lambda y_1)$, $(x_2, \lambda y_2)$ is not less than $\lambda F(x)$ for $a \leqq x_1 < x < x_2 \leqq b$, and the member passing through $(x_1, y_1 + y_1')$, $(x_2, y_2 + y_2')$ is not less than $F(x) + F'(x)$ for $a \leqq x_1 < x < x_2 \leqq b$.

Definition. A function $f(x)$ *is called a generalized convex function if the family* $F(x; \alpha, \beta)$ *satisfies the condition (1), (2), and (3), and* $f(x) \leqq F_{12}(x)$ *for all* x_1, x_2, x, *with* $a \leqq x_1 < x < x_2 \leqq b$.

For instance, (a) when $F(x; \alpha, \beta) \equiv \alpha x + \beta$, then $F(x; \alpha, \beta)$ satisfies the conditions (1), (2), and (3) and therefore the convex function in the usual sense is a generalized convex function, (b) when $F(x; \alpha, \beta) \equiv \alpha \sin \rho x + \beta \cos \rho x$ where ρ is a constant and $b - a < \frac{\pi}{\rho}$, then $F(x; \alpha, \beta)$ satisfies (1), (2), and (3) and

1) E. F. Beckenbach; Generalized convex functions, Bull. Amer. Math. Soc. Vol. 43 (1937), 363–371.