48. *On a Regular Function, whose Real Part is Positive in a Unit Circle.*

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1. Carathéodory's theory[1] of positive harmonic functions in a unit circle atracted interests of many mathematicians[1] and several proofs were given and the results were completed and now the main results stand in the following theorems. In this peper, I will give a simple proof, where the proof of Theorem 1(I) is suggested by Szasz's paper[1] and the proof of Theorem 1(II) is the same as Schur's proof[1] essentially, but in a modified form.

Teeorem 1. Let $f(z)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_nz^n$ $(a_0=\text{real})$ *be regular in* $|z|<1$. *Then* (I)(*Carathéodory*[1]*-Toeplitz*).[2] $\Re f(z)\geqq 0$ *in* $|z|<1$, *when and only when the Hermitian forms* $H_n(x)=\sum_{0}^{n}a_{\mu-\nu}x_\nu\bar{x}_\mu (a_{-\nu}=\bar{a}_\nu)$[3] *are non-negative for* $n=0, 1, 2,\ldots$. *If all* $H_n(x)$ *are non-negative and* $H_0(x),\ldots, H_{k-1}(x)$ *are positive definite and* $H_k(x)$ *is positive semi-definite, then* $f(z)$ *is of the form:*

$$f(z)=\sum_{\nu=1}^{k}\frac{r_\nu}{2}\cdot\frac{1+\epsilon_\nu z}{1-\epsilon_\nu z},\ (r_\nu>0,\ |\epsilon_\nu|=1,\ \epsilon_i\neq\epsilon_j(i\neq j)),\qquad (1)$$

where k is the rank of the infinite Hermitian matrix H:

$$H=\begin{pmatrix} a_0, & a_1, & a_1,\ldots \\ \bar{a}_1, & a_0, & a_1,\ldots \\ \bar{a}_0, & \bar{a}_1, & a_0,\ldots \\ \ldots\ldots\ldots \end{pmatrix}.$$

(II) (*I. Schur*).[1] *If we put*

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1) C. Carathéodory: Über die Variabilitatsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen. Rendiconti del circolo mat. Palermo. **32** (1911).

2) O. Toeplitz: Über die Fouriersche Entwicklung positiver Funktionen. Rendiconti del circolo mat. Palermo. **32** (1911). E. Fischer: Über das Carathéodorysche Problem. Rendiconti del circolo mat. Palermo. **32** (1911). I. Schur: Über potenzreihen, die in Innern des Einheitskreises beschränkt sind. Crelle. **147** (1917). O. Szasz: Über harmonischen Funktionen und L Formen. Math. Zeits. **1** (1918). G. Szegö Über Funktionen mit positiver Realteil. Math. Ann. **99** (1928). F. Riesz: Über ein Problem des Herrn Carathéodory. Crelle **146** (1916).

3) In this paper, $\bar{a}$ means the conjugate complex of a.