46. *On the Boundary Value of a bounded analytic Function of several complex Variables.*

By Masatsugu TSUJI.
Mathematical Institute, Tokyo University.

(Comm. by S. KAKEYA. M.I.A., May 12, 1945.)

1. Let $f(z)$ be regular and bounded in $|z|<1$. Then (i) (Fatou.)[1] $\lim f(z)=f(e^{i\theta})$ exists almost everywhere on $|z|=1$, when z tends to $e^{i\theta}$ non-tangentially to $|z|=1$. (ii) (F. and M. Riesz.)[2] If the boundary value $f(e^{i\theta})$ vanishes on a set of positive measure on $|z|=1$, then $f(z)\equiv 0$. (iii) (Szegö.)[3] If $f(z)\not\equiv 0$, then $\log|f(e^{i\theta})|$ is integrable on $|z|=1$.

We will show that an analogous theorem holds for a bounded regular function of several complex-variables.

Let $z=e^{i\theta}$, $w=e^{i\varphi}$ be points on $|z|=1$, $|w|=1$ respectively. Then the pair $(e^{i\theta}, e^{i\varphi})$ can be considered as a point on a torus $\boldsymbol{\Theta}$ $(0\leqq\theta\leqq 2\pi,\ 0\leqq\varphi\leqq 2\pi)$ and the measure of a measurable set E on $\boldsymbol{\Theta}$ is defined by

$$mE=\iint_E d\theta d\varphi, \quad \text{so that } m\boldsymbol{\Theta}=4\pi^2. \tag{1}$$

Then the following theorem holds:

Theorem 1. *Let $f(z, w)$ be regular and bounded in $|z|<1$, $|w|<1$. Then* (i) $\lim f(z, w)=f(e^{i\theta}, e^{i\varphi})$ *exists almost everywhere on $\boldsymbol{\Theta}$, when $z\to e^{i\theta}$, $w\to e^{i\varphi}$ non-tangentially to $|z|=1$, $|w|=1$ respectively.* (ii) *If the boundary value $f(e^{i\theta}, e^{i\varphi})$ vanishes on a set of positive measure on $\boldsymbol{\Theta}$, then $f(z, w)\equiv 0$.* (iii) *If $f(z, w)\not\equiv 0$, then $\log|f(e^{i\theta}, e^{i\varphi})|$ is integrable on $\boldsymbol{\Theta}$.*

Since I have proved (i) in the former paper,[4] I will prove (ii) and (iii). We remark that if $f(z, w)$ is bounded in $|z|<1$, $|w|<1$ and $|f(e^{i\theta}, e^{i\varphi})|\leqq M$ almost everywhere on $\boldsymbol{\Theta}$, then $|f(z, w)|\leqq M$ in $|z|<1$, $|w|<1$.

For, let $|z|<R<1$, $|w|<R<1$, then

$$f(z, w)=f(re^{i\theta}, \rho e^{i\varphi})$$
$$=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{f(Re^{i\theta'}, Re^{i\varphi'})(R^2-r^2)(R^2-\rho^2)d\theta' d\varphi'}{(R^2-2Rr\cos(\theta'-\theta)+r^2)(R^2-2R\rho\cos(\varphi'-\varphi)+\rho^2)}$$
$$(0\leqq r<R,\ 0\leqq\rho<R) \tag{2}$$

(1) P. Fatou: Séries trigonométriques et séries de Taylor, Acta Math. **30** (1906).

(2) F. und M. Riesz: Über die Randwerte einer analytischen Funktion. Compte rendu du quatrième congres des mathematiciens scandinaves (1916).

(3) G. Szegö: Über die Randwerte einer analytischen Funktion. Math. Ann. **84** (1921),

(4) M. Tsuji: On Hopf's ergodic theorem. Jap. Journ. Math. **19**.