37. Markoff Process and the Dirichlet Problem.

by Shizuo KAKUTANI.

Mathematical Institute, Osaka Imperial University.

(Comm. by T. TAKAGI, M.I.A., April 26, 1945)

1. The purpose of this paper is to give a general discussion of the Dirichlet problem from the standpoint of the theory of positive linear operations in a semi-ordered Banach space. It will be shown that the so-called sweeping out process of obtaining the solution of the Dirichlet problem may be observed as a kind of Markoff process[1] in the space of continuous functions.

2. Let Ω be a compact Hausdorff space. The set $C(\Omega)$ of all real-valued continuous functions $x(\omega)$ defined on Ω is a Banach space with respect to the norm:

(1) $$\|x\|=\sup_{\omega\in\Omega}|x(\omega)|.$$

$C(\Omega)$ is also an (M)-space[2] with respect to the partial ordering:

(2) $x\geqq y$ *if and only if* $x(\omega)\geqq y(\omega)$ *for all* $\omega\in\Omega$;

and $e(\omega)\equiv 1$ is the unit element of $C(\Omega)$.

3. Let D be a bounded domain in the Gaussian plane. We do not assume that D is simplv or finitely connected. Let us consider the (M)-spaces $C(\overline{D})$ and $C(\Gamma)$, where $\overline{D}$ is the closure of D and $\Gamma=\overline{D}-D$ is the boundary of D. For any $x(\zeta)\in C(\overline{D})$, let $y(\zeta)\in C(\Gamma)$ be the boundary value of $x(\zeta)$ on Γ. Then $y=A(x)$ is a bounded linear operation which maps $C(\overline{D})$ onto $C(\Gamma)$, and clearly satisfies

(3) $x\geqq 0$ *implies* $A(x)\geqq 0$,

(4) $x\equiv 1$ *implies* $A(x)\equiv 1$,

(5) $\|A(x)\|\leqq\|x\|$.

That $y=A(x)$ is an onto-mapping means the fact that, for any $y(\zeta)\in C(\Gamma)$, there exists an $x(\zeta)\in C(\overline{D})$ such that $A(x)=y$. We can take as $x(\zeta)$ any continuous extension of $y(\zeta)$ from Γ to $\overline{D}$. Such an extension, however, is not uniquely determined; but it is possible[3] to find in a concrete way a bounded linear operation $x=B(y)$ which maps $C(\Gamma)$ into $C(\overline{D})$ such that $AB(y)=y$ on $C(\Gamma)$ and further that

---

1) K. Yosida and S. Kakutani, Operator-theoretical treatment of Markoff process and the mean ergodic theorem, Annals of Math., 42(1941).

2. S. Kakutani, Concrete representation of abstract (M)-spaces and the characterization of the space of continuous functions, Annals of Math., 42(1941).

3) S. Kakutani, Simultaneous extension of continuous functions considered as a positive operation, Jap. Journ. of Math., 19(1940).