5. *On a Stochastic Integral Equation.*

By Kiyosi Itô.
Mathematical Institute, Nagoya Imperial University.
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In his note "Stochastic Integral"[1] the author has discussed an integral of the type $\int_0^t f(\tau, \omega) d_\tau g(\tau, \omega)$, where ω is a variable taking values in a probability field (Ω, P) and $g(t, \omega)$ is a normalized brownian motion on (Ω, P). This note is devoted to the investigation of a stochastic integral equation:

(1) $$x(t, \omega) = c + \int_0^t a(\tau, x(\tau, \omega)) d\tau + \int_0^t b(\tau, x(\tau, \omega)) d_\tau g(\tau, \omega),$$

which is closely related to the researches of Markoff process by many authors, especially by S. Bernstein,[2] A. Kolmogoroff,[3] and W. Feller.[4]

Theorem. Let $a(t, x)$ *and* $b(t, x)$ *be continuous in* (t, x) *and satisfy*

(2) $|a(t,x) - a(t,y)| \leqq A|x-y|$, (3) $|b(t,x) - b(t,y)| \leqq B|x-y|$, *where* $0 \leqq t \leqq 1$ *and* $-\infty < x, y < \infty$. *Then the integral cquation (1) has one and only one continuous (in t with P-measure 1) solution.*

Proof. Firstly we shall find a solution by the method of successive approximation. We define $x_k(t, \omega)$ for $k = 0, 1, 2, \dots$ as follows,

(4) $$x_0(t, \omega) \equiv c,$$

(5) $$x_k(t, \omega) = c + \int_0^t a(\tau, x_{k-1}(\tau, \omega)) d\tau + \int_0^t b(\tau, x_{k-1}(\tau, \omega)) d_\tau g(\tau, \omega);$$

the possibility of these definitions can be verified recursively if we make use of the properties of the stochastic integral shown in S.I..

By (5) we have, for $k = 0, 1, 2, \dots$.

(6) $$x_{k+1}(t, \omega) - x_k(t, \omega) = \int_0^t (a(\tau, x_k(\tau, \omega)) - a(\tau, x_{k-1}(\tau, \omega))) d\tau$$
$$+ \int_0^t (b(\tau, x_k(\tau, \omega)) - b(\tau, x_{k-1}(\tau, \omega))) d_\tau g(\tau, \omega).$$

Since $a(t, x)$ and $b(t, x)$ are continuous, $|a(t, c)|$ and $|b(t, c)|$ are bounded in $0 \leqq t \leqq 1$ by a finite upper bound, say M. Then we have

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1) These proceedings Vol. XX. No. 8. p. 519. This paper will be cited as S.I. in the following.

2) S. Bernstein: Equations différrentielles stochastiques, Actuarités Scientifiques 738.

3) A. Kolmogoroff: Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung, Math. Ann. 104, p. 415.

4) W. Feller: Zur Theorie der stochastischen Prozesse. (Existenz und Eindeutigkeitssätze.), Math. Ann. 113, p. 113.