36. A Theorem on the Poisson Integral.

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1. We will prove the following theorem,

Theorem. Let $u(z)$ ($z = re^{i\theta}$) be a harmonic function in the unit circle $|z| < 1$ and be expressed by a Poisson integral:

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\varphi}) \frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}\, d\varphi, \qquad (1)$$

where $u(e^{i\theta})$ is integrable in Lebesgue's sense, and G be any simply connected domain in $|z| < 1$.

When we map G conformally on the unit circle $|x| < 1$, $u(z)$ becomes a harmonic function $v(x)$ in $|x| < 1$.

Then $v(x)$ can be expressed by a Poisson integral of the form (1) in $|x| < 1$.

Prof. Tsuji proved this theorem in the special case in which G is bounded by a finite number of analytic curves C_i ($i = 1, \dots, k$) in $|z| < 1$ and a certain number of circular arcs on the unit circle $|z| = 1$, and the angles between any two adjoining C_i are different from zero and the angles which C_i makes with the unit circle are different from zero and π, so that C_i does not touch the unit circle.[1]

2. *Proof.* We write $u(z)$ in the form:

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\{|u(e^{i\varphi})| + u(e^{i\varphi})\} \frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}\, d\varphi$$

$$- \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\{|u(e^{i\varphi})| - u(e^{i\varphi})\} \frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}\, d\varphi. \qquad (2)$$

Since both $\{|u(e^{i\theta})| + u(e^{i\theta})\}$ and $\{|u(e^{i\theta})| - u(e^{i\theta})\}$ are positive and integrable in Lebesgue's sense, $u(z)$ can be expressed by a difference of two positive harmonic functions of the form (1), so that to prove our theorem, it suffices to prove for a positive harmonic function of the form (1), where $u(e^{i\varphi}) \geqq 0$.

We take a sequence of positive numbers, such that

$$0 < M_1 < M_2 < \cdots\cdots < M_n \to \infty$$

and define $u_n(e^{i\theta})$ as follows;

$$u_n(e^{i\theta}) = u(e^{i\theta}) \qquad \text{when } M_n \geqq u(e^{i\theta}),$$

$$u_n(e^{i\theta}) = M_n \qquad \text{when } u(e^{i\theta}) > M_n,$$

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(1) M. Tsuji, Theorems concerning Poisson integrals. Jap. Journ. Math. 7 (1930), 227—253.