19. Note on Eulerean Squares.

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1. By an Eulerean square of order n we mean a matrix $\|(a_{ij}, b_{ij})\|$, $i, j = 1, 2 \ldots\ldots, n$, formed by n^2 pairs (a_{ij}, b_{ij}) out of n symbols, say, $1, 2, \ldots\ldots, n$, so arranged that neither in a row nor in a column of the matrix one and the same symbol occurs more than once as the first term a or as the second term b of the constituent (a, b), so that the matrices $A = \|a_{ij}\|$ and $B = \|b_{ij}\|$ are the so-called Latin squares. Without loss of generality, we may assume an Eulerean square in the *normal* form, one in which the first row is made up of the constituents (i, i), $i = 1, 2, \ldots\ldots, n$.

The substitutions A_i, $i = 1, 2, \ldots\ldots, n$ of the lst. by the ith. row of a Latin square form what we call a *Latin system* of substitutions, which is characterized by the occurrence among them of all the n^2 *substitution-elements* $a \to b$, $a, b = 1, 2, \ldots\ldots, n$. We have then in connection with an Eulerean square in the normal form E, beside the *anterior* and the *posterior* Latin systems:

$$A_i = \begin{pmatrix} 1, & 2, & \ldots\ldots, & n \\ a_{i1}, & a_{i2}, & \ldots\ldots, & a_{in} \end{pmatrix}, \quad B_i = \begin{pmatrix} 1, & 2, & \ldots\ldots, & n \\ b_{i1}, & b^{i2}, & \ldots\ldots, & b_{in} \end{pmatrix},$$

the *intermediate* system

$$P_i = \begin{pmatrix} a_{i1}, & a_{i2}, & \ldots\ldots, & a_{in} \\ b_{i1}, & b_{i2}, & \ldots\ldots, & b_{in} \end{pmatrix}, \qquad i = 1, 2, \ldots\ldots, n,$$

which also make up a Latin system. Between the substitutions of the systems A, B and P the relation (i) subsists, whence also, observing that A_i^{-1} form with A_i a Latin system, the relations (ii) – (vi):

$$\begin{array}{lll} \text{(i) } A_i P_i = B_i, & \text{(ii) } B_i^{-1} A_i = P_i^{-1}, & \text{(iii) } A_i^{-1} B_i = P_i, \\ \text{(iv) } B_i P_i^{-1} = A_i, & \text{(v) } P_i B_i^{-1} = A_{i,}^{-1} & \text{(vi) } P_i^{-1} A_i^{-1} = B_i^{-1}, \end{array} \qquad (1)$$

shewing that the distinction between the extreme and the intermediate systems of an Eulerean square is relevant only to the mutual relation, not inherent in the nature of the systems themselves. Beside these, we have a *transverse* system Q consisting of the substitutions

$$Q_j = \begin{pmatrix} a_{1j}, & a_{2j}, & \ldots\ldots, & a_{nj} \\ b_{1j}, & b_{2j} & \ldots\ldots, & b_{nj} \end{pmatrix}, \qquad j = 1, 2, \ldots\ldots, n,$$

which do not make up a Latin system, but partaking with it of the property of containing among them all the n^2 substitution-element $a \to b$. P_i and Q_j contain just one substitution-element $a_{ij} \to b_{ij}$ in common, corresponding to the constituent (a_{ij}, b_{ij}) of the Eulerean square.

An Eulerean square may be conveniently represented by a set of n^2