

67. On Mercer's Theorem.

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(Comm. by. S. KAKEYA, M. I. A., Dec. 12, 1946.)

L. C. Bosanquet [1] and G. Hayashi [2] proved the absolute convergence analogue of Mercer's limit theorem. Mercer's theorem is generalized by Copson-Ferrar, S. Izumi, J. Karamata and others. In this paper we prove the absolute convergence analogue of the generalized form of Mercer's theorem. The formulation and the proof runs parallel as the author's former paper [3]

Theorem 1. If (1) $a_n > 0$, (2) $\sum_{n=1}^{\infty} 1/a_n = \infty$ and

$$(3) \quad y_n = (1 + a_n)t_n - a_n t_{n-1} \quad (n = 1, 2, \dots),$$

then $\sum_{n=1}^{\infty} |\Delta y_n| < \infty$ implies $\sum_{n=1}^{\infty} |\Delta t_n| < \infty$.

Proof. Putting $t_0 = 0$ and $a_n = 1/a_n$ and solving (3) concerning t_n , we get

$$t_n = \frac{a_1}{\prod_{\nu=1}^n (1 + a_{\nu})} y_1 + \frac{a_2(1 + a_1)}{\prod_{\nu=1}^n (1 + a_{\nu})} y_2 + \dots + \frac{a_n(1 + a_1) \dots (1 + a_{n-1})}{\prod_{\nu=1}^n (1 + a_{\nu})} y_n,$$

and then

$$\begin{aligned} \Delta t_n &= \frac{a_n}{\prod_{\nu=1}^n (1 + a_{\nu})} \{ -a_1 y_1 - a_2(1 + a_1) y_2 - a_3(1 + a_1)(1 + a_2) y_3 - \dots \\ &\quad - a_{n-1}(1 + a_1) \dots (1 + a_{n-2}) y_{n-1} + (1 + a_1) \dots (1 + a_{n-1}) y_n \} \\ &= \frac{a_n}{\prod_{\nu=1}^n (1 + a_{\nu})} \sum_{\mu=1}^n \left[\left\{ \prod_{\nu=1}^{\mu-1} (1 + a_{\nu}) \right\} (\Delta y_{\mu}) \right] \end{aligned}$$

where $y_0 = 1$ and $\prod_{\nu=1}^0 (1 + a_{\nu}) = 1$. Thus

$$\sum_{n=1}^{\infty} |\Delta t_n| = \sum_{n=1}^{\infty} \left| \frac{a_n}{\prod_{\nu=1}^n (1 + a_{\nu})} \sum_{\mu=1}^n \left[\left\{ \prod_{\nu=1}^{\mu-1} (1 + a_{\nu}) \right\} (\Delta y_{\mu}) \right] \right|$$

[1] Bosanquet, L. C., *An analogue of Mercer's theorem*, Journ. London Math. Soc., 13 (1938), 177-180.

[2] Hayashi, G., *A theorem on limit*, Tohoku Math. Journ., 45 (1939), 329-331.

[3] Sunouchi, G., *Theorems on limits of recurrent sequences*, Proc. Imperial Acad. Tokyo, 10 (1934), 4-7.