12. *On the Cartan Decomposition of a Lie Algebra.*

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Let $\mathfrak{L}$ be a Lie algebra over the field of complex numbers, $\mathfrak{H}$ and $\mathfrak{H}'$ maximal nilpotent subalgebras of $\mathfrak{L}$ containing regular elements. E. Cartan has shown for semi-simple $\mathfrak{L}$ that there exists an inner automorphism A such that $\mathfrak{H}'=A\mathfrak{H}$[1]. In this note we shall show that this theorem is valid for any, not necessarily semi-simple, Lie algebra. From this we see easily that the decomposition of a Lie algebra into the eigen-spaces of a maximal nilpotent subalgebra containing a regular element (Cartan decomposition) is unique up to inner automorphisms of $\mathfrak{L}$.

Let $\mathfrak{G}$ be the Lie group which corresponds to $\mathfrak{L}$. To every element a of $\mathfrak{L}$ corresponds a one-parameter subgroup $g(t)$ of $\mathfrak{G}$ and a is the tangent vector at the unit element to the differentiable curve $g(t)$. Extending to general Lie groups a notion familiar for matrices, we shgll denote by *exp ta* this one-parameter subgroup $g(t)$ and by *exp a* the point of parameter 1 on this curve. Further $exp\,\mathfrak{H}$ means the (local) subgroup of $\mathfrak{G}$ which corresponds to a subalgebra $\mathfrak{H}$ of $\mathfrak{L}$. If we transform the elements of the group *exp ta* by a fixed element g, we obtain a new one-parameter subgroup *exp ta′* ; the mapping $a\rightarrow A_g a=a'$ is an inner automorphism of $\mathfrak{L}$ generated by g. The mapping $x\rightarrow D_a x=[a, x]$, with a fixed, is a linear operation in $\mathfrak{L}$, which is called inner derivation of $\mathfrak{L}$. Suppose that $g=exp\,a$ and g is sufficiently near to the unit element, then $A_g=exp\,D_a$. Let us decompose $\mathfrak{L}$ by A_g into eigen-spaces:

$$\mathfrak{L}=\mathfrak{L}_1+\mathfrak{L}_\rho+\mathfrak{L}_\sigma+\ldots\ldots$$

where $\mathfrak{L}_1$, $\mathfrak{L}_\rho$, ... are the eigen-spaces for the characteristic roots, 1, ρ, ... of A_g. Here $\mathfrak{L}_1$ is a subalgebra of $\mathfrak{L}$[2].

Lemma[3]. The systems $u^{-1}g\,exp\,\mathfrak{L}_1 u$, where u runs over a neighbourhood of the unit element, contain a neighbourhood of the element g in $\mathfrak{G}$.

Proof. Let $a_1, a_2, \ldots a_s$ be a basis of the subalgebra $\mathfrak{L}_1$ and $a_{s+1}, \ldots, a_r$ a basis of $\mathfrak{L}_\rho+\mathfrak{L}_\sigma+\ldots$. Then the (local) subgroup $exp\,\mathfrak{L}_1$ is composed of all elements of the forms $exp(t_1a_1+\ldots+t_sa_s)$, where the parameters t_i are sufficiently near to zero. To prove our Lemma, it is sufficient to show that the set of elements

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1) E. Cartain, Le principe de dualité et la théorie des groupes simples et semesimples (Bull. Sc. math. t. 49, 1925). Gantmacher has given a proof in a somewhat general form. F. Gantmacher, Canonical representations of automorphisms of a complex semi-simple Lie group, (Recueil mathématique, 5(47), 1939).

2) See Gantmacher, l. c. P. 107: If g is sufficiently near to the unit element then $\mathfrak{L}_1\neq 0$.

3) Cf. Gantmacher, l. c. P. 113.