On Haar Measure of Some Groups[1)]

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In his famous paper on the theory of invariants, A. Hurwitz[2)] introduced the notion of invariant measure of group manifold. He gave an explicit expression for Haar measure of unitary unimodular groups and orthogonal groups by means of generalized polar coordinates. Afterwards H. Weyl[3)] obtained another expression. I will obtain other expressions for unitary, unitary symplectic, and orthogonal groups, using the Cayley's parametrization. Concerning unitary groups we shall prove the following:

Theorem 1. *The infinitesimal volume element $d\Omega$ of Haar measure of unitary group of n-th order is given by the following formula*

$$d\Omega = |E_n + H^2|^{-n} dh$$

where H is the Cayley's parameter of unitary matrix

$$U = (E_n + iH)(E_n - iH)^{-1}$$

and Hermitian, so that

$$\tilde{H} = H = (h_{ik}) = (a_{ik} + ib_{ik}), \ (a_{ik} = a_{ki}, \ b_{ki} = -b_{ik})$$

and dh is the product of all differentials of n parameters,

$$dh = da_{11} da_{12} \dots da_{nn} db_{21} \dots db_{n,n-1} .$$

Proof. Let U be an unitary matrix of n-th order, which is represented by Cayley's parameters as follows:

$$U = (E_n + iH)(E_n - iH)^{-1}$$

where H is a Hermitian matrix. We form the differential of U

$$dU = \{E + i(H + dH)\}\{E - i(H + dH)\}^{-1} - (E + iH)(E - iH)^{-1},$$

then we get, by left multiplication of $E - i(H + dH)$ and right multiplication of $E - iH$,

$$\begin{aligned} &\{E - i(H + dH)\} dU (E - iH) \\ &= \{E + i(H + dH)\}(E - iH) - \{E - i(H + dH)\}(E + iH) \\ &= 2i dH. \end{aligned}$$

If we neglect the terms of 2-nd order, we obtain:

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1) A brief sketch of this paper was read at the May meeting of the Mathematical Society of Japan, 1947,

2) A. Hurwitz, Ueber die Erzeugung der Invarianten durch Intergration, Göttinger Nachrichten, 1897. S. 71–90.

3) H. Weyl, Classical groups, 1939.