60. An Extension of Fokker-Planck's Equation.

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Let the possible states of a stochastic system be represented by the points $x=(x_1, \ldots, x_n)$ of the n-dimensional Riemannian space R. We denote by $P(s, x, t, E)$, $s \leqq t$, the transition probability that the state x at the time moment s is transferred into the Borel set $E \subseteqq R$ at the later time moment t. The function P will satisfy the probability conditions

(1) $$P(s, x, t, E) \geqq 0, \quad P(s, x, t, R)=1,$$

(2) $$P(s, x, s, E)=1 \text{ or } =0 \text{ according as } x \in E \text{ or } x \bar{\in} E,$$

and the Chapman-Smoluchouski's equation

(3) $$P(s, x, t, E)=\int_R P(s, x, u, dz)P(u, z, t, E), \quad s \leqq u \leqq t.$$

Let $C(R)$ be the Banach space of real-valued bounded continuous functions $f(x)$ on R with the norm $\|f\|=\sup|f(x)|$. We assume that

(4) $$(U_{st}f)(x)=\int_R P(s, x, t, dy)f(y)$$

defines a system of linear operators $\{U_{st}\}$ on $C(R)$ in $C(R)$. Then

(5) $(U_{st}f)(x)$ is non-negative with $f(x)$ and $\|U_{st}\|=1$,

(6) $U_{ss}=I$ (the identity), $U_{su}U_{ut}f=U_{st}f$.

In the special case of the temporal homogeneity

(7) $$U_{su}=T_{u-s},$$

the strong continuity in t of T_t implies the strong differentiability of T_tf for those f which are strongly dense in $C(R)$[1]:

(8) $$\frac{dT_tf}{dt}=\text{strong}\lim_{\Delta \downarrow 0}\frac{T_{t+\Delta}-T_t}{\Delta}f=AT_tf=T_tAf, \quad Af=\left(\frac{dT_tf}{dt}\right)_{t=0}.$$

In the general case, a formal extenssion of the above equation will be

(9) $$\frac{\partial U_{st}f}{\partial s}=-A_sU_{st}f.$$

It may be called as Fokker-Planck's equation corresponding to the stochastic process $P(s, x, t, E)$

The purpose of the present note is to give a possible form of the un-

1) E. Hille: Functional Analysis and Semi-groups, New York (1948). K. Yosida: On the differentiability and the representation of one-parameter semi-group of linear operators, Journal of the Math. Soc. of Japan, Vol. 1. No. 1 (1948).