19. Theory of Invariants in the Geometry of Paths. I. Determination of Covariant Differentiations.

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§ 0. In an n-dimensional space X_n with a coordinate system x^i $(i=1, 2, \dots, n)$ let us consider a system of paths of the m-th order defined by

$$(0.1) \qquad x^{(m)i}+H^i(t, x, x^{(1)}, \dots, x^{(m-1)})=0 \qquad (i=1, 2, 3, \dots, n).$$

The geometries of paths are called ordinary, intrinsic, and rheonomic geometry according to their fundamental transformation groups

$$\text{(i)} \begin{cases}\bar{x}^\alpha=\bar{x}^\alpha(x^i),\\ \bar{t}=t,\end{cases} \qquad \text{(ii)} \begin{cases}\bar{x}^\alpha=\bar{x}^\alpha(x^i),\\ \bar{t}=\bar{t}(t),\end{cases} \qquad \text{(iii)} \begin{cases}\bar{x}^\alpha=\bar{x}^\alpha(t, x^i),\\ \bar{t}=t.\end{cases}$$

Various researches were made already by many geometricians.

In this paper, we study the theory of invariants in the geometry of paths under the so-called generalized rheonomic transformation group

$$(0.2) \qquad \bar{x}^\alpha=\bar{x}^\alpha(t, x^i), \quad \bar{t}=\bar{t}(t)$$

which is a generalization of above three groups; let us assume that the functions $\bar{x}^\alpha(t, x^i)$, $\bar{t}(t)$ have continuous derivatives with respect to $t, x^1, x^2, \dots, x^n$ up to the order needed, and that none of the functional determinant $\left|\dfrac{\partial\bar{x}^\alpha}{\partial x^i}\right|$ and the derivative $\dfrac{d\bar{t}}{dt}$ vanish.

§ 1. Let v and w be two kinds of those geometric objects each of which has uniquely determined components v^i or w_j in every coordinate system x^i in X_n and t and are subject to the transformation law

$$(1.1) \qquad \bar{v}^\alpha=\sigma^p\frac{\partial\bar{x}^\alpha}{\partial x^i}v^i, \quad \bar{w}_\beta=\sigma^p\frac{\partial x^j}{\partial\bar{x}^\beta}w_j, \quad \left(\frac{1}{\sigma}=\frac{d\bar{t}}{dt}\right)$$

under (0.2). We call such a geometric object v or w respectively a *contravariant* or *covariant vector of the second kind of weight* p. A geometric quantity f which is subject to the transformation law $\bar{f}=\sigma^p f$ is called a *scalar of weight* p.

Let us denote the $n+1$ independent variables t, x^i by y^I $(I=0, 1, \dots, n)$: $y^0\equiv t$, $y^i\equiv x^i$, then the transformation group (0.2) may be written as $\bar{y}^A=\bar{y}^A(y^I)$ in the $X_{n+1}=(t)\times X_n$.