23. Wiman's Theorem on Integral Functions of Order $<\frac{1}{2}$.

By Masatsugu TSUJI.
Mathematical Institute, Tokyo University.
(Comm. by T. TAKAGI, M.J.A., May 12, 1950.)

1. Density of sets.

Let E be a measurable set on the positive x-axis and $E(a,b)$ be its part contained in $[a,b]$. We put

$$\bar{\delta}(E)=\overline{\lim_{r\to\infty}}\frac{1}{r}\int_{E(0,r)}dr \quad , \quad \underline{\delta}(E)=\underline{\lim_{r\to\infty}}\frac{1}{r}\int_{E(0,r)}dr \quad , \tag{1}$$

$$\bar{\lambda}(E)=\overline{\lim_{r\to\infty}}\frac{1}{\log r}\int_{E(1,r)}\frac{dr}{r} \quad , \quad \underline{\lambda}(E)=\underline{\lim_{r\to\infty}}\frac{1}{\log r}\int_{E(1,r)}\frac{dr}{r} \quad , \tag{2}$$

$$\bar{\lambda}^*(E)=\overline{\lim_{r/a\to\infty}}\frac{1}{\log(r/a)}\int_{E(a,r)}\frac{dr}{r}, \quad \underline{\lambda}^*(E)=\underline{\lim_{r/a\to\infty}}\frac{1}{\log(r/a)}\int_{E(a,r)}\frac{dr}{r} \quad (a\geqq 1). \tag{3}$$

We call (1) the upper (lower) density, (2) the upper (lower) logarithmic density and (3) the upper (lower) strong logarithmic density. Evidently

$$0\leqq\underline{\delta}(E)\leqq\bar{\delta}(E)\leqq 1, \quad 0\leqq\underline{\lambda}^*(E)\leqq\underline{\lambda}(E)\leqq\bar{\lambda}(E)\leqq\bar{\lambda}^*(E)\leqq 1$$

and

$$\underline{\delta}(E)+\bar{\delta}(C(E))=1, \quad \underline{\lambda}(E)+\bar{\lambda}(C(E))=1, \quad \underline{\lambda}^*(E)+\bar{\lambda}^*(C(E))=1,$$

where $C(E)$ is the complementary set of E. We shall prove:

Lemma 1. $0\leqq\underline{\delta}(E)\leqq\underline{\lambda}^*(E)\leqq\underline{\lambda}(E)\leqq\bar{\lambda}(E)\leqq\bar{\lambda}^*(E)\leqq\bar{\delta}(E)\leqq 1.$

Proof. Let $\bar{\delta}(E)=\alpha$, then for any $\varepsilon>0$,

$$\mu(r)=\int_{E(0,r)}dr\leqq r(\alpha+\varepsilon) \qquad (r\geqq r_0(\varepsilon)>1),$$

so that if $1\leqq a<r_0<r$, since $\mu(r)\leqq r$,

$$\int_{E(a,r)}\frac{dr}{r}\leqq\int_1^{r_0}\frac{dr}{r}+\int_{r_0}^{r}\frac{d\mu(r)}{r}$$
$$\leqq r_0+\left[\frac{\mu(r)}{r}\right]_{r_0}^{r}+\int_{r_0}^{r}\frac{\mu(r)}{r^2}dr\leqq r_0+1+(\alpha+\varepsilon)\int_{r_0}^{r}\frac{dr}{r}$$
$$\leqq r_0+1+(\alpha+\varepsilon)\log\frac{r}{a}.$$

If $r_0\leqq a<r$, then similarly

$$\int_{E(a,r)}\frac{dr}{r}\leqq 1+(\alpha+\varepsilon)\log\frac{r}{a}.$$