47. *Brownian Motions in a Lie Group.*

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The notion of Brownian motions has been introduced by N. Wiener [1] [2][1] in the case of the real number space (or more generally the n-space) and by P. Lévy [3] in the case of the circle. We shall here extend this notion in the case of a general Lie group.[2]

§ *1. Definition and fundamental theorems.* Let G be an n-dimensional Lie group. A random process $\pi(t)$ in G is called to be a *right* (*left*) *invariant Brownian motion* in G, if it satisfies the following five conditions **M**, **C**, **T**, **S** and **C***.

M. $\pi(t)$ is a simple Markoff process; we denote the transition probability law of $\pi(t)$ with $F(t, p, s. E)$ i.e.

$$F(t, p, s, E) = P_r\{\pi(s)\epsilon E/\pi(t) = p\}.$$

C. Kolmogoroff-Feller's continuity condition [4] [5]. For any neighbourhood U of p it holds that

$$\lim_{s\to t+0} \frac{1}{s-t} F(t, p, s, G-U) = 0$$

and the following limits exist $(1 \leqq i, j \leqq n)$

$$a^i(t, p) \equiv \lim_{s\to t+0} \frac{1}{s-t}\int_U (x^i - x^i_0)\, F(t, x_0, s, dx),$$

$$B^{ij}(t, p) \equiv \lim_{s\to t+0} \frac{1}{s-t}\int_U (x^i - x^i_0)(x^j - x^j_0)\, F(t, x_0, s, dx),$$

where (x^i) is a local coordinate defined on U and (x^i_0) is the coordinate of p.

T. temporal homogeneity. $F(t, p, s, E) = F(t+\tau, p, s+\tau, E)$.

S. spatial homogeneity.
right invariance $F(t, p, s, E) = F(t, pr, s, Er)$.
(left invariance $F(t, p, s, E) = F(t, lp, s, lE)$.)

C* continuity. Almost all sample motions[3] are continuous.

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1) The numbers in [ ] correspond to those in the the references at the end of this paper.

2) Prof. K. Yosida has obtained a similar result in making use of his operator-theoretical method. See the preceding article.

3) In the analytical theory of probability a random motion is represented by a motion depending on a probability parameter. Any motion for each parameter value is called to be a sample motion.